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Interrelations between String Theory and neutron scattering on 1D ferromagnets

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Abstract

Integrable models of spin chains appear in many varied applications, and in this thesis two of those are studied. One is the determination of anomalous dimensions of operators in the conformal field theory(CFT) $\mathcal{N} = 4$ Super Yang-Mills(SYM). The other is how their solution is used to make theoretical predictions about the behavior of 1-dimensional magnets and compare those to experiments done using neutron scattering.

The solution of the CFT is used to test an interesting conjecture which relates string theory and SYM - the AdS/CFT-correspondence.

Specifically, the calculations presented is the complete diagonalization of the dilatation operator of SYM to 1-loop order, which shows that the Hilbert space of the operator to 1-loop is extended to include multiple-trace operators. Finally, this leads to a postulation of the new ideas needed to make neutron measurements on multiple-trace chains.

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Chapter 1

Introduction

Some of the most fascinating aspects of physics are models or descriptions that goes beyond a specific problem or system. This constitutes one of the main aspects of studying physics - to find new ways to relate different aspects of nature in a mathematical way, and thereby learn new secrets about the world. In this specific case, the two different areas being related to each other are nano-scale structures, and theoretical high-energy physics. The link between these two far-removed subjects is an integrable quantum-mechanical model called the Heisenberg spin chain, which describes particles with spin sitting on a chain. The solution to the model was found in the beginning of the last century by Hans Bethe[1], and turned out to be integrable. Therefore much effort has been put into realizing these one-dimensional structures by chemists and solid-state-physicists, and some progress in comparing with theoretical calculations based on the solution by Bethe has been made[2]. As will be explained in chapter 3, this spin chain has been found to directly correspond to operators in the conformal field theory(CFT) $\mathcal{N} = 4$ Super Yang-Mills (SYM) (for planar diagrams), and thereby make it possible to determine the scaling dimensions of the operators easily using the known solution.

The reason that the scaling dimensions, and their quantum corrections, are interesting, is a conjecture called the AdS/CFT correspondence. Based on the review [3], and especially [4] and [5] therein, the conjecture is explained in chapter 2. It relates the masses of string-states in IIB super string theory on AdS which is Anti-de Sitter, a symmetric space with constant negative curvature, to the anomalous dimensions of operators in the conformal field theory $\mathcal{N} = 4$ SYM. During the search for ways to prove the conjecture, the integrable structure previously mentioned was found, so this is where we turn back to the question of what this could mean.

In this case, we have found that something which was originally solved as a problem of 1D chains of atoms with spin, can now be used to find anomalous dimensions of operators in a conformal field theory. How come the mathematics turns up in two so different places - could it be a hint at some underlying

pattern which we have yet to find? This question is the reason for the current directions of research, where many are trying to find integrable structures also at higher loop order(when including higher order quantum corrections). Unfortunately, they have not yet been successful, and therefore in this work, the perturbative approach is used so solve the problem of the scaling dimensions.

Also, even though the spin chain solution is used both for finding anomalous dimensions and for describing physical spin chains, a final link to experiment is needed. Although the model is integrable which means it is possible to solve it algebraically, it is still a challenge to determine measurable quantities from it.

The thesis is organized in the following way: After this introduction, a couple of other areas where field theories are related to solid-state physics are discussed. Then we begin with the main topic of the thesis - the AdS/CFT correspondence, which will be explained qualitatively in chapter 2. With the reasons found in this chapter as a basis, in chapter 3 some basics of conformal field theory are introduced, which leads to the derivation of the dilatation operator. After a description of the planar limit and integrability in chapter 4, comes an introductory chapter on neutron scattering in chapter 5, which concludes in the result which shows that we need to find the result of applying the spin operator to eigenstates of the spin system. Therefore in chapter 6, we determine the eigenstates of the dilatation operator in a useful basis, and find that some of them are double- and triple-trace operators. The last chapter concludes the findings of the thesis.

1.1 Quantum field theories in other solid-state phenomenon

The idea to use the theoretical methodology of conformal field theories in other areas than high energy physics is not new. Since CFT's are so general in nature, effects in condensed matter physics have also been described theoretically by formulating CFT's of quasi-particles. The technique has proven to be able to explain phenomena previously not understood, and it is even possible to test some predictions experimentally. The American scientist Subir Sechdev and colleagues have been using the correspondence to aid them in developing theoretical descriptions of so called 'strange metals' and have successfully arrived at new predictions which have been proven experimentally [6]. They find that the electrons in the metals can be described by a quantum field theory and a gauge field A_μ in much the same way as quantum electrodynamics. The field represents the mutual effect that the spins have on each other, and the two states of the ferromagnet are represented by the 'photon' state and the 'Higgs' state of the resulting quantum field theory.

Chapter 2

The AdS/CFT correspondence

In this chapter, the AdS/CFT correspondence is motivated through a rough description of the string theory developments that was the basis - and a more stringent description of the field theoretic ideas that in the end led to the development of a clear correspondence. Describing all string theory development is beyond the scope of this work, so only the most groundbreaking developments will be sketched, which leads to the developments important for the discovery of the AdS/CFT correspondence.

In the late 1960's, the first developments of string theory were begun, with the aim to find a description of the many mesons found at CERN at the time[7]. String theory is based on the idea that instead of point-like particles, we have tiny one-dimensional strings that are either open or closed. In spite of seeming like a very beautiful, and also productive, idea because the resulting theory includes gravity with the rest of the forces, it was found that the theory only was definable on a world of 26 dimensions. The super-string formalism reduced this number to 10, but it still contained super-luminous particles, tachyons, that introduced instabilities in the theory. On top of this, quantum chromodynamics was developed, which removed the need for a theory of the mesons in the standard model.

Although most work in string-theory gradually stopped, some were still researching ways to relate the idea of strings to the field-theoretical picture. In 1974, t' Hooft[8] had realized that when going to N colors in a field theory it turns into a string theory. This can be shown very beautifully using the known diagrammatic expansion to the field theories. In fig. 2.3 we have an example of two diagrams in some general field theory. There is a planar one and a non-planar one, as can be recognized by the lines crossing each other in the non-planar one. If we now imagine letting these diagrams have a topology other than the plane. The surface they are drawn on takes the role of the worldsheet of a string. Then we see that the planar diagram can be drawn on a space which has the topology of a sphere, but the non-planar one needs a

space with topology like a torus to be drawn correctly because of the crossing lines.

In the mid-90's Dirichlet branes, or D-branes for short, brought string theory back to gauge theory. D-branes are a kind of membranes of various internal dimensionalities contained in theories of closed superstrings. The ends of the strings can attach to the branes, which results in particle-like excitations, which combined with the t'Hooft limit of the field theories further strengthened the relation between string theory and field theories.

2.0.1 Formulating the correspondence

Using this realization Maldacena[9] formulated the AdS/CFT correspondence in 1997 as follows. The theory of the D-branes needs to be specified more precisely to be able to see the basis for the correspondence. The theory we will be looking at is IIB string-theory on flat, 10D minkowski-space, in a background of N so called D-branes that are stacked. The theory is illustrated in fig. 2.1. This results in two kinds of strings - one is a closed string in the free space, and the other is a open strings extending between the branes. The open and closed strings each have a low-energy limit, and it is this limit which we will now take a closer look at.

2.0.2 Strings as excitations

In the low-energy limit, the closed strings gain an effective Lagrangian equal to that of IIB super strings. The open strings on the other hand have a low-energy representation with an effective Lagrangian equal to that of $\mathcal{N} = 4$ SYM[10]. The theory effectively decouples in the low-energy limit into free gravity in the bulk and SYM on the border. This is an example of a very well tested principle, that string theory on a curved background of d+1 dimensions corresponds to a field theory on the d-dimensional border of the d+1-dimensional background. This is called the holographic principle, and the AdS/CFT correspondence is an example of this idea.

2.0.3 D-branes as excitations

Looking in another way at the theory, the D-branes can also be represented as excitations in the theory, with mass and charge. Then if we take the low-energy limit, it is clear that there are two ways to achieve that: One is for the branes to have very long wavelengths and thereby achieve low energy, and the other is when they approach $r = 0$[10] and the observer sits at infinity. Again the theory decouples at low energies, since the long-wavelength states will be concentrated at $r \to \infty$ and the ones approaching $r = 0$ does not have the energy to climb the existing gravitational barrier to escape to the asymptotic

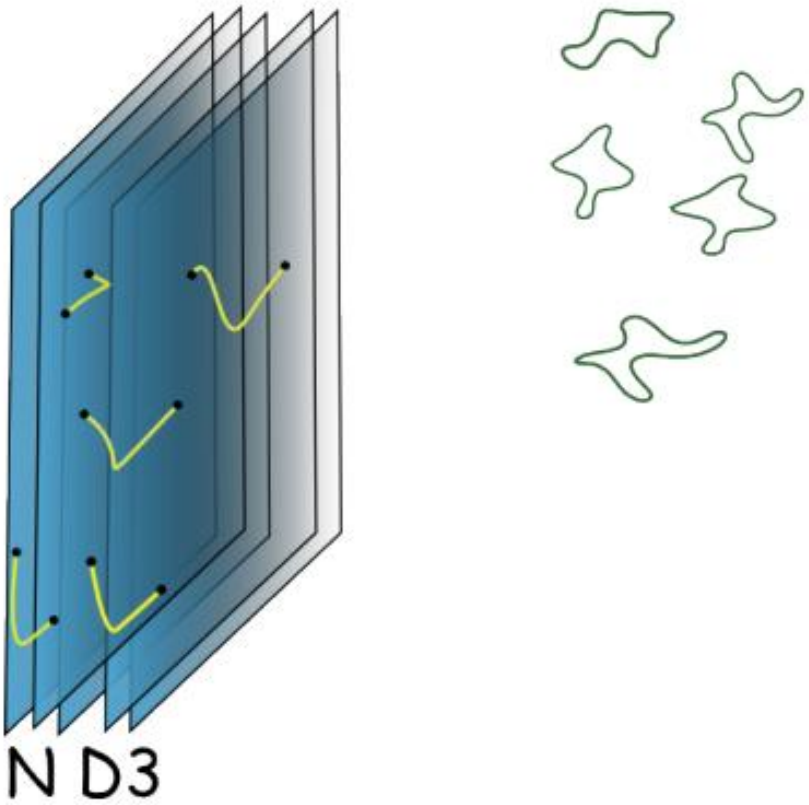


Figure 2.1: A schematic illustration of the string theory on a background of N D3 branes. There are both open strings (the ones attaching to the branes) and closed free strings.

Figure 2.2: Here we see how two different views of the same theory each decouples into two theories when going to the low energy limit. We end up with two times two decoupled theories - one is the same in both views, so therefore we conclude that the other two must be equivalent.

region. The final point is that the geometry of space in the asymptotic region is given as $AdS_5 \times S^5$.

The conclusion is that the same theory, when viewed in two different ways, decouples in the low energy limit to two theories, where one of the two new theories are the same in both views(as illustrated in fig. 2.2). A logical conclusion would be, that the two other decoupled parts should then be equal to one another. Therefore Maldacena (and co.) were led to conjecture:

$\mathcal{N} = 4$ *Super-symmetric Yang-Mills theory in 3+1 dimensions is dual to IIB super-string theory on a background of* $AdS_5 \times S^5$

The conjecture also requires the following relation between the parameters of the theories:

$$g_{YM} = g_s, \qquad R^4 = 4\pi g_s N \alpha'^2 = 4\pi\lambda\alpha'^2. \tag{2.1}$$

Since then, much research has been done in trying to specify the conjecture further. The conclusion of the AdS/CFT correspondence is that the mass of string-states on $AdS_5 \times S^5$ corresponds to the anomalous dimensions of operators in the CFT $\mathcal{N} = 4$ SYM[11].

$$\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})} \rangle_{CFT} = \mathcal{Z}_{string}[\phi(\vec{x}, z)|_{z=0} = \phi_0(\vec{x})]. \tag{2.2}$$

This relation is not possible to prove generally yet, but in a specific limit it is possible. The verification that for dual observables these quantum numbers indeed match is called the spectral problem of AdS/CFT

2.1 Limits

To be able to test the correspondence directly, some limits to the theories are needed. The one which is used in this work, which has been found to be very effective, is the previously mentioned t'Hooft limit. It is defined by introducing the t'Hooft coupling λ and require it to be constant while N goes to infinity:

$$N \to \infty \quad \text{with} \quad \lambda = g_{YM}^2 N = \text{constant}, \tag{2.3}$$

where g_{YM} is the coupling constant, and N is the number of colours of the CFT. This translates to the following in string-theory terms:

$$N \to \infty \quad \text{with} \quad \lambda g_s = \frac{\lambda}{N} \to 0. \tag{2.4}$$

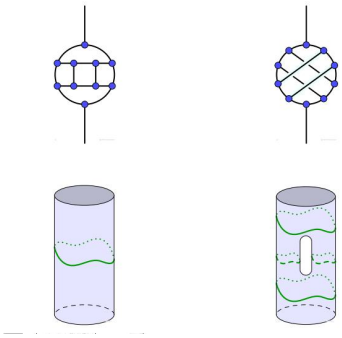


Figure 2.3: The non-planar diagram corresponds to interacting strings

Here g_s is the string coupling constant. We see that in this limit the effective coupling constant of the CFT becomes the t'Hooft coupling λ. The string theory becomes non-interacting, so this limit is a weakened form of the AdS/CFT correspondence which relates planar SYM to weakly coupled strings on $AdS_5 \times S^5$. In this limit integrability appears on both sides of the correspondence, which makes it possible to make tests of its validity, which shows good agreement[3].

The splitting and joining of strings are introduced when we take non-planar Feynman diagrams into consideration[4]. The non-planar solution to the dilatation operator is seen to have the effect of splitting the traces of the operators. Therefore the focus of this thesis will be on the non-planar part of the dilatation operator. The reason is that the least well-known question in relation to string-theory is the one describing interacting strings, which means splitting or joining strings.

One problem with the consideration of non-planar diagrams is that the framework of integrability still is not directly applicable to the description of these diagrams. In reference [4] most, if not all, of the current ideas about how to use integrability in the attempt to diagonalize non-planar Hamiltonians are reviewed, and the conclusion is that the methods not are there yet. It seems that the most useful method of diagonalization still is normal quantum mechanical perturbation theory.

Chapter 3

Conformal field theory

Since the AdS/CFT correspondence relates conformal field theories with stringtheory, and the relation is between anomalous dimensions of the CFT and mass of the string states, some knowledge about the CFT in question is needed. This is what this chapter will go through, which ends with a derivation of the dilatation operator for $\mathcal{N} = 4$ SYM.

Conformal transformation are scale transformations. So theories that are conformally invariant are scale invariant, their physics at different scales is linked.

3.1 $\mathcal{N} = 4$ Super Yang-Mills theory

The first Yang-Mills theory was proposed in 1954 by Chen Ning Yang and Robert Mills. This theory was later developed into $\mathcal{N} = 4$ Super Yang-Mills which is used in the AdS/CFT correspondence. Originally it did not receive that much interest since the particles it describes are massless. Instead of particles we have local operators, and instead of mass we have something called the scaling dimension.

The quantum field theory used in the AdS/CFT correspondence is a very special one. It is a non-Abelian gauge theory like QCD, and exists in 3+1 dimensions with gauge group $SU(N)$. It is special because it has the highest possible amount of symmetry possible for a renormalizable conformal theory. This explains why it is called N=4 Super-Symmetric Yang-Mills.

All the symmetries of the theory together form the super-conformal group $PSU(2,2|4)$. $SO(4,2) \times SO(6)$ is the bosonic part of the this global symmetry group and describes the bosonic field content of the theory. The group ($SO(6)$) corresponds to R-symmetri, which is an internal symmetry among the scalar fields of the theory. $SO(4,2)$ corresponds to the conformal group, which contains 4 special conformal transformations (K_μ) that preserve angles between lines, one generator of scale transformations or dilatations (D), 4 translations

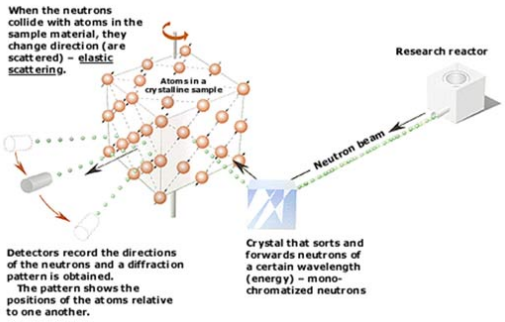


Figure 3.1: this is not the figure - only a standin - This is the field content of $\mathcal{N} = 4$ SYM

(P_μ) as well as 6 Lorentz-transformations of rotations and boost $(L_{\mu\nu})$*. Dilatations:

$$\mathcal{D} : x_\mu \to \lambda x_\mu. \tag{3.1}$$

Special conformal transformations:

$$K_\mu : x_\mu \to \frac{x_\mu + a_\mu x^2}{1 + 2x^\nu a_\nu + a^2 x^2}. \tag{3.2}$$

These generators satisfy the following commutation relations to achieve a closed algebra[5]:

$$\begin{array}{ll} [P_\mu, P_\nu] = 0 & [P_\mu, K_\nu] = 2i(L_{\mu\nu} - \eta_{\mu\nu}\mathcal{D}) \\ [L_{\mu\nu}, \mathcal{D}] = 0 & [L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \\ [K_\mu, \mathcal{D}] = iK_\mu & [P_\mu, L_{\nu\rho}] = i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu) \\ [P_\mu, \mathcal{D}] = iP_\mu & [L_{\mu\nu}, K_\rho] = -i(\eta_{\mu\rho}K_\nu - \eta_{\rho\nu}K_\mu) \\ [K_\mu, K_\nu] = 0 & [\mathcal{D}, \mathcal{D}] = 0 \end{array}$$

where $\eta_{\mu\nu}$ is the ... The field content of the theory is portrayed in fig. 3.1. There are 6 scalars Φ_a which are in the fundamental representation of the R-symmetry, then there are 8 fermionic fields in the Weyl spinor representation, or 4 in the Dirac spinor representation: $\Psi_{\alpha A}$, $\Psi^{\bar{A}}_{\dot{\alpha}}$ that are spinors both in Lorentz symmetry and in R-symmetry. Finally there are the gauge fields A_μ that only have Lorentz vector symmetry.

Here the indices are defined as follows: The R-symmetry vector index is $a = 1, 2, \cdots, 6$ and the R-symmetry spinor indices are $A, \bar{A} = 1, 2, 3, 4$. Similarly for the Lorentz symmetry we have $\mu = 0, 1, 2, 3$ which is the Lorentz vector index and $\alpha, \dot{\alpha} = 1, 2$ that are the Lorentz spinor indices. [12]

To extend the theory to be supersymmetric, the generators of supersymmetry, the supercharges Q^A_α, $Q^{\bar{A}}_{\dot{\alpha}}$ are introduced. When commuting these with

*Details of these transformations are to be found in Appendix C

the conformal generators K_μ, we can introduce further, the super-conformal charges S^A_α and $\bar{S}^{\dot{A}}_{\dot{\alpha}}$[13]. Their commutation and anti-commutation relations with the generators of the conformal algebra are(now subduing all indices):

$$\begin{array}{ll} [D,Q] = \frac{-i}{2}Q & [D,S] = \frac{i}{2}S \\ [K,Q \approx S & [P,S] \approx Q \\ \{Q,Q\} \approx P & \{S,S\} \approx K \\ \{Q,S\} \approx M + D + R & \end{array}$$

This extends the algebra to the N = 4 super conformal algebra $PSU(2,2|4)$. The symmetries of the theory gives us the following Lagrangian[14]:

$$\mathcal{L}_{YM}(x,g) = Tr(\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}\mathcal{D}^\mu\Phi^n\mathcal{D}_\mu\Phi_n - \frac{1}{4}g^2[\Phi^m,\Phi^n][\Phi_m,\Phi_n] \tag{3.3}$$

$$+ \dot{\Psi}^a_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu\mathcal{D}^\mu\Psi_{\beta\alpha} - \frac{1}{2}ig\Psi_{\alpha a}\sigma^{ab}_m\varepsilon^{\alpha\beta}[\Phi^m,\Psi_{\beta\alpha}] - \frac{1}{2}ig\dot{\Psi}^a_{\dot{\alpha}}\sigma^m_{ab}\varepsilon^{\dot{\alpha}\dot{\beta}}[\Phi_m,\dot{\Psi}^b_{\dot{\beta}}]). \tag{3.4}$$

We see that it only has terms of dimension 4, and also that the only free parameters of this theory is the coupling g, and N. The high level of symmetry in this theory is special, but not the most original trait of the theory. That is the fact that the conformal symmetry survives quantization which has the special consequence that the β-function equals zero[5]:

$$\beta = \mu\frac{\partial g}{\partial\mu} = -\frac{g^3_{YM}}{16\pi^2}\left(\frac{11}{3}N - \frac{1}{6}\sum_i C_i - \frac{1}{3}\sum_j \tilde{C}_j\right) = 0, \tag{3.5}$$

where the first sum is over all scalars with casimirs C_i and the second is over all Weyl fermions with casimirs C_j. Since all the fields in $\mathcal{N} = 4$ SYM are in the adjoint representation of the group, all the casimirs are equal to N, which gives us the result.

That the β-function is zero means that we only need to do renormalization on the wavefunctions of the fields, which specifically results in a quantum-mechanical correction to the scaling dimension of the operators of the theory.[14]

3.2 The dilatation operator

The observables of any theory are the correlation functions, such as for example the two-point function:

$$\langle\mathcal{O}(0)\mathcal{O}(x)\rangle, \tag{3.6}$$

and in a conformal theory such as the $\mathcal{N} = 4$ SYM, they are defined from conformal symmetry, and are given by:

$$\langle\mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle = \frac{C}{|x-y|^{2\Delta}}. \tag{3.7}$$

Here Δ's are the conformal dimensions of the operators, which specify the scaling behavior of the operators. These have quantum corrections called the anomalous dimensions:

$$\Delta = \Delta_0 + \gamma. \tag{3.8}$$

Since we are looking at observable operators in the theory, they need to be gauge invarian. In general the gauge transformation on some local field $\chi(x)$ has the following form[5]:

$$\chi(x) \to \chi(x) + [\chi(x), \varepsilon(x)]. \tag{3.9}$$

More fields can be generated by taking the covariant derivative of this field like:

$$\mathcal{D}_\mu \chi(x) \equiv \partial_\mu \chi(x) - [\mathcal{A}_\mu(x), \chi(x)]. \tag{3.10}$$

Only one kind of operators satisfy this criteria for gauge invariance:

$$\mathcal{O} = \mathrm{Tr}(\chi(x)_1 \chi(x)_2 \chi(x)_3 \dots \chi(x)_L), \tag{3.11}$$

where the trace is over the N color indices and L is the number of fields in the trace, ie. the length of the operator. Since the superconformal charges $S^A_\alpha(\bar{S}^{\dot{A}}_{\dot{\alpha}})$ increase(decrease) the dimension of the operator it is used on by $\frac{1}{2}$, there is a group of operators whose dimension is already as low as possible which will therefore fulfill:

$$[S^a_\alpha, \mathcal{O}(0)] = 0. \tag{3.12}$$

The operators that fulfill this are called primary operators, and the operator in eq. 3.11 satisfies this equation, and therefore their anomalous dimensions are protected. They are called BPS-operators. Since we are interested in the anomalous dimensions, we choose some specific operators that are not primary - we inserts some "impurities" so that they will have anomalous dimensions:

$$\mathcal{O} = \mathrm{Tr}(\chi(x)_1 \chi(x)_2 \Xi(x)_3 \dots \chi(x)_{L-1} \Xi(x)_L), \tag{3.13}$$

where the number of impurities $\Xi(x)$ is low compared to the lenght of the operator L. This is called the BMN limit, and when the number of impurities are specifically 2, then the operators are called BMN operators.

When calculating the anomalous dimensions in $\mathcal{N} = 4$ SYM, it is found that these are given as the eigenvalues of general dilatation operators or matrices. To simplify the calculation, we are staying in the $SU(2)$ subsector of the theory. This means that the fields constituting our operators are given as

$$\begin{aligned} Z &= \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2) \\ \phi &= \frac{1}{\sqrt{2}}(\Phi_3 + i\Phi_4), \end{aligned} \tag{3.14}$$

and their conjugates($\bar{\phi}$ and $\bar{Z}$)where the Φ_i's are the scalar fields of the theory. Inserting this in the expression for the BMN operators, we have the final expression for the operators used here:

$$\mathcal{O}(x) = \mathrm{Tr}(Z \dots Z\phi Z \dots Z\phi Z \dots Z). \tag{3.15}$$

Dilatations are defined in general for any theory. They are given as the transformation that "scales" any operator $\mathcal{O}(x)$ by a definite factor: [10]

$$\mathcal{O}(x) \to \mathcal{O}(x\lambda) \cdot \lambda^{\Delta} \tag{3.16}$$

where Δ is the constant of scaling. For the operators consisting of scalar fields, at tree level when no quantum corrections are taken into account, the scaling dimension is given as the classical scaling dimension. This is the length of the operator since the classical scaling dimension of scalars is equal to one:

$$\begin{aligned} \Delta_0(\phi) = 1 \to \\ \hat{D}_0\mathcal{O}(x) = L. \end{aligned} \tag{3.17}$$

Which gives us this result for the planar dilatation operator:

$$\hat{D}_0 = \mathrm{Tr}(\phi_m\hat{\phi}_m) \tag{3.18}$$

In essence, the dilatation operator just counts the number of fields in the operator. Going to 1-loop level, the first corrections from quantum field theory is taken into account, and diverting diagrams appear. Therefore the operators have to renormalized[12]:

$$\mathcal{O}_{ren} = \mathcal{O}_{bare}\left(\frac{\mu}{\Gamma}\right)^{\gamma(g^2)}, \tag{3.19}$$

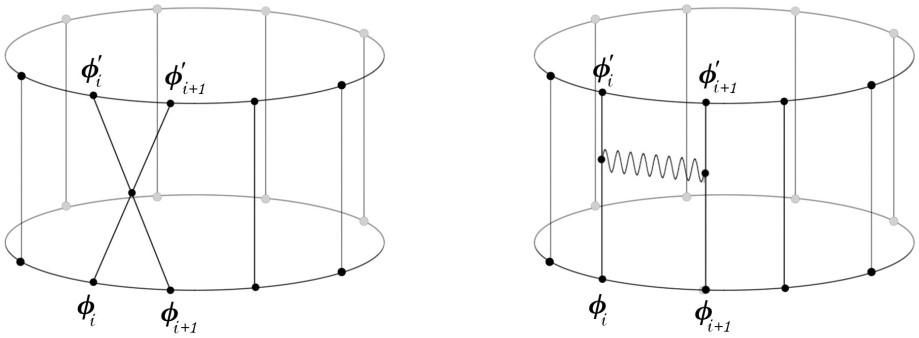
where Γ is a energy cutoff we choose for the theory and μ is the mass after renormalization. $\gamma(g^2) = \gamma g^2 + \gamma g^4 + \dots$ are called the anomalous dimensions. Including this redefinition, now when a dilatation with the effect of $\mu \to \mu\lambda$, the resulting renormalized operator is:

$$\mathcal{O}_{ren} \to \mathcal{O}_{ren}\lambda^{\Delta_0}\left(\frac{\mu\lambda}{\Gamma}\right)^{\gamma(g^2)}, \tag{3.20}$$

Similarly, we make a perturbative expansion of the dilatation operator:

$$\hat{D} = \hat{D}_0 + \lambda(\hat{D}_1 + (\mathrm{ordersof}\lambda^2). \tag{3.21}$$

The $\hat{D}_1$ contribution corresponds to the 1-loop contributions and orders of higher λ are not taken into account because we are only going 1-loop order.



(a) Four-scalar interaction (b) Gluon exchange (c) Self e

Figure 3.2: The diagrams contributing to the 1-loop dilatation operator.

3.3 Determining the 1-loop dilatation operator

So to determine the full dilatations-operator at one loop, we need to sum the contributions from all 1-loop diagrams. This calculation was done the first time in 2002 by Minahan and Zarembo[11], and in fig. **??** the diagrams are shown. We see that there are 3 vertices that contribute to 1-loop diagrams:

- the merging of two neighboring scalar propagators
- gluon exchange between two neighboring scalars
- self-energy corrections to the scalar propagator, a fermion loop or a gluon propogator

They diagrams are shown in fig. 3.2. To find the specific expression for the dilatation operator, we need to derive the contribution from each of these diagrams, where when calculating the contributions from each diagram, we divide by the non-contracted diagram in order to normalize the result.

First we identify the vertices responsible for each diagram. They are found from the Lagrangian eq. (3.3), and are given as:

- scalar four-point vertex:

$$U_{4s} = -\frac{1}{2g_{YM}^2}\mathrm{Tr}[\phi_i, \phi_j][\phi_i, \phi_j]. \tag{3.22}$$

- scalar-gluon vertex

$$U_{sg} = -\frac{2i}{g_{YM}^2}\mathrm{Tr}\partial_\mu\phi_i[A_\mu, \phi_j]. \tag{3.23}$$

- fermion-scalar vertex

$$U_{fg} = -\frac{i}{g_{YM}^2}\mathrm{Tr}\bar{\psi}\Gamma_i[\phi_i, \psi]. \tag{3.24}$$

Using these we will now calculate the contribution from each of the diagrams, following the treatment in [15] closely. Further details can be found there..

3.3.1 Scalar four-point vertex

Since there are no contracted fields inside this diagram, only the contractions in the vertex contributes. Since the the fields will always be contracted either with the operator at 0 ($\mathcal{O}(0) = \mathcal{O}^-$) or at x ($\mathcal{O}(x) = \mathcal{O}^+$) it is useful to split the scalar[15]:

$$\phi_i = \phi_i^+ + \phi_i^- = i^+ + i^-, \tag{3.25}$$

where we have introduced in the usual way the new notation i^+ and i^-. With this definition, the allowed contractions are specified by this delta-function

$$\langle i^+, i^- \rangle = \delta^{ij}, \tag{3.26}$$

which allows us to rewrite the interaction:

$$U_{4s} = -\frac{1}{g_{YM}^2}(\mathrm{Tr}[i^+, i^-][i^+, i^-] + \mathrm{Tr}[i^+, i^-][i^-, i^+] + \mathrm{Tr}[i^+, i^+][i^-, i^-]). \tag{3.27}$$

Finishing the calculation, the coupling constant given by the Feynman integral in $D = 4 - 2\varepsilon$ is

$$\left(\frac{g_{YM}^2}{8\pi^2 x^2}\right)^{L-2-L}\left(\frac{g_{YM}^2}{8\pi^2}\right)^4 \int \frac{dz}{(x-z)^4 z^4} = \frac{g_{YM}^4 \Lambda}{32\pi^2}, \tag{3.28}$$

where Λ is:

$$\Lambda = \log x^{-2} - \left(\frac{1}{\varepsilon} + \gamma + \log\pi + 2\right), \tag{3.29}$$

With this, the final result is:

$$V_{4s} = -(:\mathrm{Tr}[i^+, i^-][i^+, i^-] + \mathrm{Tr}[i^+, i^-][i^-, i^+] : + : \mathrm{Tr}[i^+, i^+][i^-, i^-] :)\frac{g_{YM}^2 \Lambda}{16\pi^2}. \tag{3.30}$$

Where the normal ordering symbols have been added for clarity - the contractions are only meant to apply to fields outside the effective vertex.

3.3.2 Gluon exchange

By extending the exponent of the action so that we have two vertices between gluons and scalars, we get the following expression for the potential of the gluon exchange diagram:

$$U_{sg} = -\frac{4}{g_{YM}^4}\mathrm{Tr}\partial_\mu\phi_i[A_\mu,\phi_i]\mathrm{Tr}\partial_\nu\phi_{i+1}[A_\nu,\phi_{i+1}]. \tag{3.31}$$

Rewriting the traces:

$$\begin{aligned}\mathrm{Tr}\partial_\mu\phi_i[A_\mu,\phi_i] &= \mathrm{Tr}\partial_\nu\phi_i(A_\nu\cdot\phi_i-\phi_i\cdot A_\nu)\\ &= \mathrm{Tr}A_\nu(\phi_i\partial_\nu\phi_i-\partial_\nu\phi_i\phi_i) = \mathrm{Tr}A_\mu[\phi_i,\partial_\mu\phi_i]\text{'}\end{aligned}$$

we get this new expression for the vertex after performing the gluon contractions:

$$U_{sg} = \frac{4}{g_{YM}^4}\mathrm{Tr}[\phi_i,\partial_\mu\phi_i][\phi_j,\partial_\mu\phi_j]. \tag{3.32}$$

A factor of $\frac{1}{N}$ times traces of single commutators appear, but they are equal to zero because of the cyclicity of the trace. Inserting the new notation from eq. (3.25) this is changed to:

$$U_{sg} = \frac{8}{g_{YM}^4}\mathrm{Tr}[i^+,i^-][j^+,j^-], \tag{3.33}$$

which combined with the Feynman integral for the coupling constant gives the final result:

$$V_{sg} = \frac{g_{YM}^2(2+\Lambda)}{32\pi^2}\mathrm{Tr}[i^+,i^-][j^+,j^-]. \tag{3.34}$$

3.3.3 Self-energy diagrams

Self energy to the scalar propagator consists of two diagrams - one with a gluon propagator, and one with a fermion loop, represented in fig. 3.2c.

Gluon propagator

The gluon propagator is found using the expression we found above eq. (3.33) and contracting with two additional scalars:

$$U_{gss} = \frac{32}{g_{YM}^4}\mathrm{Tr}[i^+,i^-][j^+,j^-] \to N\mathrm{Tr}i^+i^-. \tag{3.35}$$

Again some factors of $\frac{1}{N}$ appear, but they are equal except they have opposite signs, so they add up to zero.

Fermion loop

Re-writing the fermion loop to include two fermions:

$$U_{sff} = \frac{1}{g_{YM}^4}\mathrm{Tr}\bar{\psi}[\phi_i, \psi]\mathrm{Tr}\psi[\bar{\psi}, \phi_i]. \tag{3.36}$$

We see that after doing the contractions of the fermions, again we get the same structure as (3.33), which we add with the equal contribution from the gluon propagator above to get the collective contribution to the dilatation operator from self-energy diagrams (after doing the Feynman integral, all the Feynman integrals of this section can for example be found in ref. [16]):

$$V_{se} = \frac{g_{YM}^2(\Lambda + 1)}{8\pi^2} N\mathrm{Tr}i^+i^-. \tag{3.37}$$

3.4 Cancellations

Now to find the form of the dilatation operator in the SU(2) sector, we just have to add the contributions from the 3 diagrams that we found above. The details of the calculation will not be shown, but the result is that the contributions from the gluon propagator and the self-energy diagrams cancel. We are left with:

$$D_2 = -\frac{g_{YM}^4\Lambda}{16\pi^2}\left(:\mathrm{Tr}[i^+, i^+][j^-, j^-]: + \frac{1}{2}:\mathrm{Tr}[i^+, i^-][j^+, j^-]:\right) - \frac{g_{YM}^2}{8\pi^2}:\mathrm{Tr}([Z, \phi][\hat{Z}, \hat{\phi}]):, \tag{3.38}$$

where the normal ordering symbols means that the operators only act on the state which the dilatation operator is being applied to - not on the fields inside the operator itself. The last equality sign comes when inserting the $SU(2)$ scalars Z and ϕ and their conjugates $\hat{Z}$ and $\hat{\phi}$. With this we have the final expression for the dilatation operator to 1-loop order in $\mathcal{N} = 4$ SYM. Later the eigenstates and eigenvalues of this operator are determined. But first we will now take a look at the planar limit of this operator, where the integrable structure the Heisenberg spin chain has been found.

Chapter 4

Spin chains - the planar limit of $\mathcal{N} = 4$ SYM

Since the AdS/CFT correspondence relates the anomalous dimensions of the CFT with the masses of string states in IIB string theory, there anomalous dimensions have to be determined. Also, it has been found that integrable structures appear on the field theory side in the specific limit, the t'Hooft limit. Therefore this limit will now be investigated in more detail, and the integrable structure explained. This derivation is essentially the same as the one at the end of chapter 3, the only difference being that we now stay in the planar sector of the theory, since this is where the integrability appears. As before, we begin with the two-point correlator defined by conformal symmetry to be:

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta}}, \tag{4.1}$$

where again we have that Δ is given as $\Delta = \Delta_0 + \gamma$ and γ is the anomalous dimension. Making the reasonable assumption that the quantum contribution to the scaling dimension is much smaller than the classical scaling dimension: $\gamma \ll \Delta_0$, we can make the expansion:

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \approx \frac{1}{|x-y|^{2\Delta_0}}(1-\gamma \ln \Lambda^2|x-y|^2), \tag{4.2}$$

where Λ is some cutoff energy. We see that what we are looking for is another expression for the correlator of two operators in SU(2) from which it will be possible to read off the anomalous dimension, γ. In the calculation in chapter 3, the whole dilatation-operator - including both planar and non-planar diagrams was determined. Now to select only the planar diagrams, the t'Hooft limit[8] is used, as explained previously in chapter 2.1. This lets $N \to \infty$. In this limit, planar diagrams dominate, since they achieve an extra N^2. We can realize

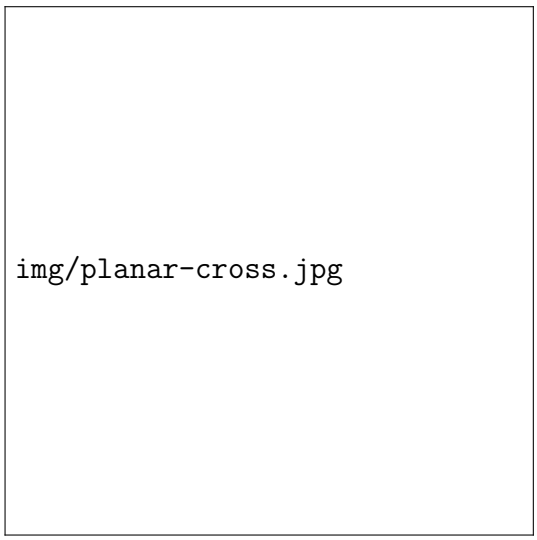


Figure 4.1: Image (a) and (b) represents planar diagrams and (c) is nonplanar, since in order to cross the lines, they have to be raised out of the plane.

this by looking closer at the contractions in the diagrams of fig. 4.1. The contractions in these diagrams can be represented like this[5]:

$$\ldots \delta_A^{A'} \delta_{A'}^{A} \delta_B^{B'} \delta_{B'}^{B} \delta_C^{C'} \delta_{C'}^{C} \cdots = \ldots N^3 \ldots, \tag{4.3}$$

for the planar diagrams, and

$$\ldots \delta_A^{A'} \delta_{B'}^{A} \delta_B^{C'} \delta_{A'}^{B} \delta_C^{B'} \delta_{C'}^{C} \cdots = \ldots N \ldots, \tag{4.4}$$

Therefore it is safe to ignore all non-planar diagrams in the following. To determine the form of the operator responsible for the anomalous dimensions, the two-point function must be determined directly from the Feynman diagrams so that it is possible to compare it with the expansion in eq. (4.2). As we found in part 3.2, to one-loop level, only one diagram contributes, namely the 4-scalar interaction. We then find that the correlator (in the $SO(6)$ sector) can be expressed as:

$$\langle \mathcal{O}_1(x) \bar{\mathcal{O}}_2(y) \rangle \approx \frac{1}{|x-y|^{2\Delta_0}} \left(1 - \frac{\lambda}{16\pi^2} \ln \Lambda^2 |x-y|^2 \sum_{\ell=1}^{L} (C - 2P_{\ell,\ell+1} + K_{\ell,\ell+1}) \right) \tag{4.5}$$

$$\times \delta_{i_1}^{j_1} \cdots \delta_{i_L}^{j_L} + cycles. \tag{4.6}$$

If we compare this equation with eq. (4.2), then it is clear that the anomalous dimension corresponds to the eigenvalues of the operator:

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{\ell=1}^{L} (C - 2P_{\ell,\ell+1} + K_{\ell,\ell+1}), \tag{4.7}$$

P is the discrete shift operator, which exchanges flavor indices on sites l and $l+1$ and K contracts the indices. K is the trace operator, and they are defined like

Now the problem of determining the anomalous dimensions has been reduced to the problem of finding the eigenvalues of this operator. If we take a second look at that operator, but go to SU(2) space, we see that it is given by:

$$\Gamma_{SU(2)} = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{L} (1 - P_{\ell,\ell+1}). \tag{4.8}$$

This can be rewritten to be a function of spin operators instead of the permutation operator. The permutation-operator is represented in matrix notation in the following way:

$$\hat{P}(x \otimes y). \tag{4.9}$$

For a 2-site spin chain, this translates to the simple expression:

$$\hat{P} \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix} = \begin{pmatrix} y_1 x_1 \\ y_1 x_2 \\ y_2 x_1 \\ y_2 x_2 \end{pmatrix}. \tag{4.10}$$

From this we can deduce that the matrix-representation of $\hat{P}$ is:

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.11}$$

For simplicity, to represent the spin operators the σ^i Pauli matrices are used, which have the matrix representation:

$$\sigma_l^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_l^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \sigma_l^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.12}$$

Taking the matrix products:

$$\sigma_l^1 \otimes \sigma_n^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \sigma_l^2 \otimes \sigma_l^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{4.13}$$

$$\sigma_l^3 \otimes \sigma_l^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.14}$$

which put together gives:

$$\sigma_l^1 \otimes \sigma_l^1 + \sigma_l^2 \otimes \sigma_l^2 + \sigma_l^3 \otimes \sigma_l^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.15}$$

It is now obvious that there is a direct relation between P and σ, which is:

$$1 - P = \frac{1}{2}(1 - \bar{\sigma} \otimes \bar{\sigma}). \tag{4.16}$$

And using this we can rewrite eq. (4.8) to:

$$\frac{\lambda}{8\pi^2} \sum_{\ell=1}^{L} (\frac{1}{2} - 2\vec{S}_\ell \cdot \vec{S}_{\ell+1}). \tag{4.17}$$

This operator is recognized as the Hamiltonian for a very well-known quantum mechanical system called the Heisenberg spin chain, or the XXX Heisenberg chain. One of the reasons for its reputation is that it is algebraically solvable, which was showed the first time in 1931 by Hans Bethe[1]. The result that the anomalous dimensions are described by a Hamiltonian for a spin chain, leads us to make the following identification:

BMN operators of the form $O_n^J = \mathrm{Tr}(Z^{J_1}\phi Z^{J-J_1})$ *of length* J *plus two impurities, correspond to spin chains of length* $L = J + 2$ *that are rotationally invariant*

,

with it is natural to introduce the notation: $Z = \downarrow$ and $phi = \uparrow$ since we are in the SU(2) sector of the conformal field theory.

With this solution, finding the anomalous dimensions has been reduced to the problem of finding the eigenfunctions and eigenvalues of this Hamiltonian. After a short digression about integrability in general, and what it means for the Heisenberg Hamiltonians solvability, determining these eigenstates and eigenvalues is done in section 4.2.

4.1 Integrability

The result of the previous chapter showed us that the reason the planar limit is special, is that the anomalous dimensions are given as the eigenvalues of the Hamiltonian of the Heisenberg $XXX_{\frac{1}{2}}$. The reason this Hamiltonian is special is that it is integrable, and therefore this section elaborates on what integrability is, and what it means specifically for this case.

If a system is integrable it means that it is possible to solve it analytically, which is rare when considering quantum field theory or even normal quantum mechanical systems. One of the reasons for the growing interest in the AdS/CFT correspondence is that integrable systems have been found on both sides of the correspondence, which makes it possible to make reasonable comparisons between them. It has also made it possible to make new interesting conclusions about the two theories individually.

Defining what integrability means in quantum mechanical systems can be a challenge when wanting a strict mathematical definition. Therefore it is best to start somewhere with a well-defined solution, which for example could be a classical system with a Hamiltonian H[p,q], where p and q are the positions and conjugate momenta. Being a classical system, the evolution of the system must obey the Hamilton equations:

$$\dot{q}_j = \{H, q_j\} = \frac{\partial H}{\partial p_j} \tag{4.18}$$

$$\dot{p}_j = \{H, p_j\} = -\frac{\partial H}{\partial q_j}, \tag{4.19}$$

where we have used the Poisson bracket defined by:

$$\{f, g\} = \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}. \tag{4.20}$$

It is now an easy task defining what integrability means for a classical system like this:

$$\{H, I_j\} = 0, \quad \{I_i, I_j\} = 0 \quad \forall \quad i, j. \tag{4.21}$$

In other words, there has to exist an operator which poisson-commutes with the Hamiltonian of the system, and with itself, then the system is integrable. To transfer this to integrability for quantum mechanical system, we quantize in the usual way:

$$I_j \to \hat{I}_i, \quad \{\ ,\ \} \to \frac{i}{\hbar}[\ ,\ ], \tag{4.22}$$

which gives us that a quantum mechanical system described by the Hamiltonian $\hat{H}$ is integrable if:

$$[\hat{H}, \hat{I}_j] = 0, \quad [\hat{I}_i, \hat{I}_j] = 0 \quad \forall \quad i, j. \tag{4.23}$$

We see that the integrability-statement for a quantum-mechanical system implies that there exists an operator which can be diagonalized simultaneously with the Hamiltonian.

But this requirement is not enough, since in reality the operators have to be algebraically independent, and not all commuting operators are also algebraically independent, as for example $\hat{H}$ and $\hat{H}^2$. Unfortunately it is not possible to make a more precise statement of integrability, but if we constrain ourselves to looking at systems of scattering particles, then it is possible to define integrability more precisely.

A system of diffracting particles could for example be a Heisenberg spin chain, which is the case in the AdS/CFT correspondence. Therefore we will choose that system to show integrability on - a two-magnon spin chain of length L. If we make no other assumptions about the system, then all we know is that the two magnons will have to satisfy energy and momentum conservation. This specifies the wavefunction to be[14]:

$$\Psi(x_1, x_2) = e^{i(p_1 x_1 + p_2 x_2)} + S(p_1, p_2) e^{i(p_1 x_2 + p_2 x_1)}. \tag{4.24}$$

This is the famous Bethe anzats, and $S(p_1, p_2)$ is the S-matrix, which describes how the particles when the diffraction occurs. Now, in general, a system of interacting particles will have the following wavefunction:

$$\begin{aligned} \Psi(x_1, x_2, ..., x_n) \approx \sum \Psi(\sigma) exp(i(p_{\sigma(1)} x_1 + ... + p_{\sigma(n)} p_n)) \\ + \Psi_{diffractive}(x_1, x_2, ..., x_n). \end{aligned} \tag{4.25}$$

This should also apply to our case of the Heisenberg Hamiltonian. We see that the wavefunction splits into a non-diffractive and a diffractive part. The σ's refer to the different permutations of the particles. Since we know that the non-diffractive part of this wavefunction is solved using the Bethe eq.s, we have the following condition for integrability:

$$\forall\ n:\ \Psi_{diffractive}(x_1, x_2, ..., x_n) = 0. \tag{4.26}$$

The Hamiltonian that describes the spin chain used in this work is:

$$\hat{H} = \lambda \sum_{n=1}^{L} (1 - P_{n,n+1}), \tag{4.27}$$

in which we can see that there is no diffractive part. This means that the Heisenberg spin chain is integrable, and we do not need to use any complicated methods to determine its integrability. If, on the other hand, we had been working with a spin-chain with more different particles than two, then the integrability would not have been obvious, and the methods of integrability would have been needed to do the calculations.

Figure 4.2: A scematic of the imagined spin chain

4.2 Determining the anomalous dimensions

Previously in chapter 4 we found that the anomalous dimensions of planar operators in $N = 4$ SYM can be represented by the eigenvalues of the XXX Heisenberg spin chain Hamiltonian. We also learned that this spin chain is integrable, so we are now ready to find the solutions and thereby the anomalous dimensions. The solution will be based on the Hamiltonian as a function of the permutation-operator, as found in the first section of chapter 4:

$$\hat{H} = \lambda \sum_{n=1}^{L} (1 - P_{n,n+1}). \tag{4.28}$$

To find the eigenvalues we solve the Schrödinger equation:

$$\hat{H}|\Psi\rangle = E|\Psi\rangle, \tag{4.29}$$

where $|\Psi\rangle$ are the eigenstates. We have one condition, which is that the trace has to be cyclic, and we therefore have that $\Psi_{m+L} = \Psi_m$ as seen on fig 4.2. The Hilbert space of a chain of length L is given as a tensor product of L local quantum spaces:

$$\mathcal{H} = V \otimes V \otimes V \otimes V \otimes \cdots \otimes V, \tag{4.30}$$

and since we are working in the SU(2) sector of the theory, the spin can only take two values, which defines V to be:

$$V = \mathbb{C}^2, \tag{4.31}$$

which is spanned by two states, the up-spin($|\uparrow\rangle$) and down-spin($|\downarrow\rangle$). The groundstate is given by:

$$\hat{H}|\uparrow\rangle \otimes |\uparrow\rangle \otimes \ldots |\uparrow\rangle \otimes |\uparrow\rangle = 0. \tag{4.32}$$

Down spins represent excitations, so to create a spin chain which corresponds to the BMN state we add two down spins. Then the eigenvector will be given as a superposition of all possible locations of the excitations:

$$|\Psi\rangle = \sum_{l_1,l_2=1}^{L} \Psi(p_1, p_2)|\uparrow\uparrow \ldots \uparrow\downarrow\uparrow \ldots \uparrow\downarrow\uparrow \ldots \uparrow\uparrow\rangle. \tag{4.33}$$

From inserting this into the Schrödinger equation eq. (4.29) we get two equations for $|\Psi\rangle$, one for $l_1 + 1 < l_2$ and one for $l_1 + 1 = l_2$. These two

results correspond to two different situations - one where the excited spins are positioned far away from each other, and one where they are each others neighbors. Our solutions has to fulfill both of the requirements represented by the two equations. Here we notice a restriction which is inherent in the Heisenberg spin chain - the only interactions we are taking into account are the ones between neighboring spins (we are only considering interactions between neighboring spins), which is the reason the solutions are divided into only two groups. The two equations are:

$$\begin{aligned} l_1 + 1 < l_2 : E\Psi(l_1, l_2) =& 2\Psi(l_1, l_2) - \Psi(l_1 - 1, l_2) - \Psi(l_1 + 1, l_2) \\ &+ 2\Psi(l_1, l_2) - \Psi(l_1, l_2 - 1) - \Psi(l_1, l_2 + 1) \end{aligned}$$

$$l_1 + 1 = l_2 : E\Psi(l_1, l_2) = 2\Psi(l_1, l_2) - \Psi(l_1 - 1, l_2) - \Psi(l_1 + 1, l_2). \tag{4.34}$$

using the Bethe ansatz from eq. (4.24):

$$\Psi(l_1, l_2) = e^{ip_1 l_1 + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 l_1}, \tag{4.35}$$

we will determine an expression for the energy, E. We insert for $\Psi(l_1, l_2)$ in (4.34) we get this expression:

$$\begin{aligned} E(e^{ip_1 l_1 + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 l_1}) =& 2(e^{ip_1 l_1 + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 l_1}) \\ &- (e^{ip_1 (l_1 - 1) + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 (l_1 - 1)}) \\ &- (e^{ip_1 (l_1 + 1) + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 (l_1 + 1)} \\ &+ 2(e^{ip_1 l_1 + ip_2 l_2} + S(p_2, p_1) e^{ip_1 l_2 + ip_2 l_1}) \\ &- (e^{ip_1 l_1 + ip_2 (l_2 - 1)} + S(p_2, p_1) e^{ip_1 (l_2 - 1) + ip_2 l_1}) \\ &- (e^{ip_1 l_1 + ip_2 (l_2 + 1)} + S \end{aligned} \qquad (p_2, p_1) e^{ip_1 (l_2 + 1) + ip_2 l_1}$$

To make sense of all this mess, I will start by isolating all the factors not dependent on S on both sides of the equation, and rewrite it in the following way:

$$\begin{aligned} E e^{ip_1 l_1 + ip_2 l_2} =& 2e^{ip_1 l_1 + ip_2 l_2} - e^{ip_1 (l_1 - 1) + ip_2 l_2} - e^{ip_1 (l_1 + 1) + ip_2 l_2} \\ &+ 2e^{ip_1 l_1 + ip_2 l_2} - e^{ip_1 l_1 + ip_2 (l_2 - 1)} - e^{ip_1 l_1 + ip_2 (l_2 + 1)} \\ &\downarrow \\ E e^{ip_1 l_1 + ip_2 l_2} =& e^{ip_1 l_1 + ip_2 l_2} (4 - e^{ip_1} - e^{ip_2} - e^{-ip_1} - e^{-ip_2}) \end{aligned}$$

Now we do the same with the parts dependent on S, and we find

$$\begin{aligned} ES(p_2,p_1)e^{ip_1l_2+ip_2l_1} &= 2S(p_2,p_1)e^{ip_1l_2+ip_2l_1} \\ &\quad - S(p_2,p_1)e^{ip_1l_2+ip_2(l_1-1)} - S(p_2,p_1)e^{ip_1l_2+ip_2(l_1+1)} \\ &\quad + 2S(p_2,p_1)e^{ip_1l_2+ip_2l_1} - S(p_2,p_1)e^{ip_1(l_2-1)+ip_2l_1} \\ &\quad - S(p_2,p_1)e^{ip_1(l_2+1)+ip_2l_1} \\ &\quad \updownarrow \\ ES(p_2,p_1)e^{ip_1l_2+ip_2l_1} &= e^{ip_1l_2+ip_2l_1}(4 - e^{ip_1} - e^{ip_2} - e^{-ip_1} - e^{-ip_2}) \end{aligned}$$

Putting these two results together then gives us the following equation:

$$\begin{aligned} (e^{ip_1l_1+ip_2l_2} + S(p_2,p_1)e^{ip_1l_2+ip_2l_1})(E - 4 - e^{ip_1} - e^{ip_2} - e^{-ip_1} - e^{-ip_2}) = 0 \\ \Psi(l_1,l_2)(E - 4 - e^{ip_1} - e^{ip_2} - e^{-ip_1} - e^{-ip_2}) = 0 \end{aligned}$$

Where we recognize the wavefunction on the right. Since the solution where the wavefunction is zero is trivial, we find that the energy is given by:

$$\begin{aligned} E &= 4 + e^{ip_1} + e^{ip_2} + e^{-ip_1} + e^{-ip_2} \\ &= 4(\sin^2(\frac{p_1}{2}) + \sin^2(\frac{p_2}{2})). \end{aligned} \tag{4.36}$$

Here we are working with a notation where the λ of the field theory has been absorbed into the definition of the energy:

$$E \to \frac{E}{\lambda}. \tag{4.37}$$

Now we have found the energy for the case of large separation, or $l_2 > l_1 + 1$. We see that the result corresponds to the sum of the energies of the individual excitations. If we now go to the case where the two magnons are situated next to each other, we had that the Schrödinger equation combined with our known spin-Hamiltonian gives us eq. (4.34):

$$E\Psi(l_1,l_2) = 2\Psi(l_1,l_2) - \Psi(l_1-1,l_2) - \Psi(l_1+1,l_2). \tag{4.38}$$

Again we will be using the Bethe ansatz eq. (4.35) which gives us:

$$\begin{aligned} E(e^{ip_1l_1+ip_2l_2} + S(p_2,p_1)e^{ip_1l_2+ip_2l_1}) &= 2(e^{ip_1l_1+ip_2l_2} + S(p_2,p_1)e^{ip_1l_2+ip_2l_1}) \\ &\quad - (e^{ip_1(l_1-1)+ip_2l_2} + S(p_2,p_1)e^{ip_1l_2+ip_2(l_1-1)}) \\ &\quad - (e^{ip_1(l_1)+ip_2(l_2+1)} + S(p_2,p_1)e^{ip_1(l_2+1)+ip_2(l_1+1)} \end{aligned}$$

Now we just need to concentrate all the factors containing an S on one side, and the rest on the other:

$$\begin{aligned} S(p_2,p_1)(Ee^{ip_1l_2+ip_2l_1} - e^{ip_1l_2+ip_2l_1} + e^{ip_1l_2+ip_2(l_1-1)} + e^{ip_1(l_2+1)+ip_2(l_1+1)}) \\ = -Ee^{ip_1l_1+ip_2l_2} + 2e^{ip_1l_1+ip_2l_2} - e^{ip_1(l_1-1)+ip_2l_2} - e^{ip_1l_1+ip_2(l_2+1)} \end{aligned}$$

Inserting $l_2 = l_1 + 1$

$$S(p_2, p_1) = \frac{2e^{ip_1 l_1 + ip_2(l_1+1)} - e^{ip_1(l_1-1)+ip_2(l_1+1)} - e^{ip_1 l_1 + ip_2(l_1+2)} - Ee^{ip_1 l_1 + ip_2(l_1+1)}}{Ee^{ip_1(l_1+1)+ip_2 l_1} - 2e^{ip_1(l_1+1)+ip_2 l_1} + e^{ip_1(l_1+1)+ip_2(l_1-1)} + e^{ip_1(l_1+2)+ip_2 l_1}}$$

Now we only need to insert the result we found previously for the energy eq. (4.36), and we get the following expression for $S(p_2, p_1)$:

$$S(p_2, p_1) = -\frac{1 + e^{ip_1+ip_2} - 2e^{ip_2}}{1 + e^{ip_1+ip_2} - 2e^{ip_1}}. \tag{4.39}$$

We can now plug this result into eq. (4.35) to find an expression for the superposition functions:

$$\Psi(l_1, l_2) = e^{ip_1 l_1 + ip_2 l_2} - \frac{1 + e^{ip_1+ip_2} - 2e^{ip_2}}{1 + e^{ip_1+ip_2} - 2e^{ip_1}} \cdot e^{ip_1 l_2 + ip_2 l_1}. \tag{4.40}$$

To continue our determination of the eigenvalues, we need an expression for p_1 and p_2. To find this, we first note that because we need cyclic symmetry, the total momentum must be equal to zero:

$$\begin{gathered} T|\Psi\rangle = |\Psi\rangle \\ \updownarrow \\ T = e^{iP} \\ \updownarrow \\ P = 0 \rightarrow p_2 = -p_1 \end{gathered}$$

. where T is the translation operator, and P is the total momentum. If we insert this in our expression for the S-matrix it simplifies significantly:

$$\begin{aligned} S(p_2, p_1) &= -\frac{1 + e^{ip_1+ip_2} - 2e^{ip_2}}{1 + e^{ip_1+ip_2} - 2e^{ip_1}} \\ &= -\frac{1 + 1 - 2e^{-ip_1}}{1 + 1 - 2e^{ip_1}} \\ &= - - \frac{e^{-i\frac{p_1}{2}}}{e^{i\frac{p_1}{2}}} \frac{1 - e^{-ip_1}}{1 - e^{-ip_1}} \\ &= e^{-ip_1}, \end{aligned} \tag{4.41}$$

which gives us a much more simple expression for ψ:

$$\Psi(l_1, l_2) = e^{ip_1(l_1-l_2)} + e^{-ip_1} \cdot e^{ip_1(l_2-l_1)}. \tag{4.42}$$

Finally we have the Bethe equations:

$$\begin{aligned} S(p_1, p_2) &= e^{ip_1 L}, \\ S(p_2, p_1) &= e^{ip_2 L}. \end{aligned} \tag{4.43}$$

If we insert the $S(p_2, p_1)$ from eq. (4.39), we get an expression for p_1:

$$\begin{aligned} e^{ip_1} &= e^{ip_1 L} \\ e^{ip_1(L-1)} &= 1, \end{aligned} \tag{4.44}$$

$$p_1 = \frac{2\pi n}{L-1}, n = 0, 1, 2, .., \underset{\downarrow}{\frac{J}{2}}. \tag{4.45}$$

Inserting this in the expression for the eigenstate, we have solved the problem, and found the eigenstate only as a function of the length of the chain:

$$\Psi(l_1, l_2) = e^{i\frac{2\pi n}{L-1}(l_1 - l_2)} + e^{-i\frac{2\pi n}{L-1}} \cdot e^{i\frac{2\pi n}{L-1}(l_2 - l_1)}. \tag{4.46}$$

But we need to rewrite the solution a bit, since we are dealing with operators in $\mathcal{N} = 4$ SYM. In this theory all the operators are given as traces of fields. We would like this to be expressed clearly in our solution. We will also find that this seemingly complication of the expression in the end will result in a much simpler way to calculate the eigenvalues and eigenstates. To achieve this, the variables l_1 and l_2 are discarded in favor of the combination $p = l_2 - l_1$. We see that p is the separation of the two excitations, which is of course rotation invariant. This change means that now the sums in our expression for the eigenstate has to run over p instead:

$$|\Psi\rangle = \sum_{p=0}^{J} \Psi(p) | \uparrow \cdots \uparrow\downarrow\uparrow \cdots \uparrow\downarrow\uparrow\rangle. \tag{4.47}$$

Here it is important to note the change in the ranges of the sum - p must be summed from 0 instead of from 1, which was the case for l_1 and l_2. And since p cannot be larger than the whole length minus the two impurities, the sum must stop at $L - 2$ which is defined as J, the number of Z-fields. We also need to exchange the position-dependent superposition state $| \uparrow \cdots \uparrow\downarrow\uparrow \cdots \uparrow\downarrow\uparrow\rangle$ with something rotation invariant. The directly equivalent operator in the CFT is defined in this way[16]:

$$\mathcal{O}_p^J = \mathrm{Tr}(\phi Z^p \phi Z^{J-p}). \tag{4.48}$$

Now the only thing left is determining the new $\Psi(p)$. To do this we take the result we already found for the eigenfunctions:

$$\Psi(l_1, l_2) = e^{i\frac{2\pi n}{L-1}(l_1 - l_2)} + e^{-i\frac{2\pi n}{L-1}} \cdot e^{i\frac{2\pi n}{L-1}(l_2 - l_1)}, \tag{4.49}$$

and change the variables l_1 and l_2 to the variable p. We keep the variable p_1 at the moment because it simplifies the expressions a lot:

$$\begin{aligned}\Psi(p) &= e^{ip_1(-p)} + e^{-ip_1} \cdot e^{ip_1 p} \\ &= e^{i\frac{-p_1}{2}} \left(e^{-ip_1(p-\frac{1}{2})} + e^{ip_1(p-\frac{1}{2})} \right) \\ &= e^{i\frac{-p_1}{2}} \cos\left(p_1(p - \frac{1}{2}) \right). \end{aligned} \tag{4.50}$$

Inserting $p_1 = \frac{2\pi n}{L-1}, n = 0, 1, 2, .., \frac{J}{2}$ into this expression and reducing we get:

$$\Psi(p) = 2e^{i\frac{p_1}{2}} \cos\left(\frac{\pi n}{1+J}(2p-1) \right), \tag{4.51}$$

where the factor in front of the cosine can be removed by normalization of the wavefunction so that we get the result:

$$\Psi(p) = \cos\left(\frac{\pi n}{1+J}(2p-1) \right). \tag{4.52}$$

Inserting the new expression for Ψ and the rotation-invariant operators $\mathcal{O}_p^J$ in the expression for the eigenstates eq. (**??**) we get:

$$\mathcal{O}_n^J = \sum_{p=l_2-l_1=1}^{L-1} \Psi(p) \mathcal{O}_{p-1}^J. \tag{4.53}$$

Here the definition of the eigenstates has also been reformulated to be position independent, and since a mode number "n" enter the solution for p_1 that label is added as well. The factor of $\frac{1}{J+1}$ in front comes from the requirement that the eigenstates be orthonormal to each other. Now, we would like the sum to go from 0, and to do that we change variables $\tilde{p} = p - 1$:

$$\begin{aligned}\mathcal{O}_n^J =& \frac{1}{J+1} \sum_{\tilde{p}=0}^{J} \cos\left(\frac{\pi n}{1+J} 2(\tilde{p}+1) - 1 \right) \mathcal{O}_{p-1}^J \\ =& \frac{1}{J+1} \sum_{\tilde{p}=0}^{J} \cos\left(\frac{\pi n}{J+1}(2\tilde{p}+1) \right) \mathcal{O}_{\tilde{p}}^J. \end{aligned} \tag{4.54}$$

Now that it is obvious that the spin chain is a model of operators in planar $\mathcal{N} = 4$ SYM, we need to find a way to measure the quantities that are interesting in this context. The observables of a system of this kind are the dynamical structure factors, which are fourier transforms of the dynamical spin-spin correlation functions [2], and they will be described in more detail in the following chapter 5 on neutron scattering. Unfortunately, the interest

of this work are the 1-loop solutions, which are not described in this simple way, they do not even have an integrable solution. They are described by the solution found previously in section that also include non-planar diagrams 3.2 which are described by a perturbation to the planar solution. As we found earlier, this perturbation results in eigenstates that are multi-trace, which significantly enlarges the Hilbert-space of the operator compared to the usual Heisenberg Hamiltonian. What this means in relation to the dynamical structure factors used to relate the Heisenberg chain to experiments will be addressed in the last part of the next chapter after an introduction to neutron scattering in general.

Chapter 5

Neutron scattering

In this chapter, basics of and thereafter more advanced topics in neutron scattering will be explained.

5.1 Basics of neutron scattering

The basic idea of neutron scattering is to use neutrons as the probe of microscopic structures. The field started a short while after the first nuclear reactors became operational since this made free neutrons accessible for the use in experiments. C. Shull and B. Brookhouse did the first real neutron scattering experiments in the late 1940s, for which they got the Nobel prize in 1994.

When doing a neutron-scattering experiment, there is going to be some kind of sample, which will be bombarded by neutrons. Some of these neutrons will then scatter of the sample and be collected in some form of detector. An illustration of a very schematic setup for a neutron-scattering experiment is seen in fig. 5.1

As an introduction, the five foremost reasons for using neutrons in experiment instead of for example x-rays are[17]:

- **Energy and wavelength.** The wavelength of thermal neutrons is around 1 angstrom, which means that it is perfect for probing atomic structures. Further, the energy of the neutrons is similar to the excitation energies of solids. Therefore it is possible to obtain simultaneous information about both the structure and dynamics of solids

- **Isotopes and light elements** The value usually measured in neuron scattering experiments called the neutron scattering cross section varies randomly between elements, and even between isotopes of the same element. Therefore neutron scattering can be used to target specific isotopes.

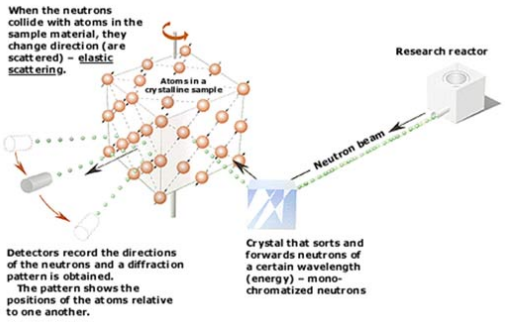


Figure 5.1: Here a basic neutron scattering setup is schematized. First, the neutrons are sent through a moderator to remove some of their energy, after which specific wavelengths can be chosen in the monochromator if needed. Then the neutrons are directed onto the sample, and the scattered neutrons are measured in the detector.

- **Quantitative experiments** Because the interaction between the neutron and other atoms is small, it is possible to probe into the samples. The week interaction also means that high-order effects play a smaller role, and therefore it is easier to compare with theoretical predictions.
- **Penetration** Neutrons can easily penetrate large experimental setups, which means that it poses no problems using for example dilution refrigerators.
- **Magnetism** Even though the neutron has neutral charge, it still has a magnetic moment due to the charges of the quarks of which it is made. It can therefore be used to probe the magnetic structure of solids.

As is hopefully obvious from this listing, neutron scattering has many and varying applications. The one that is important for this work is the last point on the list - magnetism. The reason is that the Heisenberg spin chain consists of magnetic particles, which means that we need to utilize the magnetic properties of the neutron to probe them. Therefore the description of neutron scattering coming in the two next sections will focus on an introductory description of the basics of neutron scattering, and a more detailed description of neutron scattering on magnetic systems.

To do these experiments, the most basic requirements are as follows: We need a source of neutrons, a sample, and a detector.

5.1.1 Sources

At the time of the first experiments, the only neutrons available were the ones created in nuclear reactors. The neutrons are produced by the fission processes

Figure 5.2: The two ways of making neutrons.

in the reactor, that can for example be processes of this kind:

$$^{235}\mathrm{U} + \mathrm{n} \rightarrow \mathrm{D}_1 + \mathrm{D}_2 + \mathrm{Q} + \text{appr. 2-3 neutrons}, \tag{5.1}$$

where the D's are various daughter nuclei. This method is still in use today, although now only with specially designed nuclear reactors with compact cores, which increases the neutron density.

In the 1970's another way of creating free neutrons was taken into use. This method is called spallation, which describes how it works quite literally. The idea is that if we bombard a heavy atom with a neutron, then it will get excited, and to remove some of its energy it will radiate neutrons. The two ways of creating neutrons is depicted in fig. 5.2.

5.1.2 Moderators

Since the neutrons produced in these neutron sources usually have energies around MeV, a lot of energy needs to be removed in order to be able to perform experiments on meV systems. This is done in the moderator. To remove energy from the neutrons, they need to collide with other particles. And for the energy-transfer to be efficient, the particles they collide with should have a mass comparable to the neutrons. This could for example be a proton. Therefore, most moderators consist of normal water, at room temperature. Since the collisions of the neutrons with the water-molecules essentially result in the neutrons coming into thermal equilibrium with the water, the temperature of the water is important. The corresponding temperature for a specific energy of neutrons is related as:

$$E = m_n v^2 \frac{1}{2} = k_B T_{equi}. \tag{5.2}$$

Since the temperature is important, other moderators are sometimes used, for example liquid hydrogen or solid methane.

5.1.3 Detectors

The detector is usually based on a nuclear reaction where the neutron is absorbed by a nucleus which then changes into another nucleus. An often used

example is when helium-3 captures a neutron and changes into a hydrogen nucleus and a proton:

$$^3\mathrm{He} + \mathrm{n} \rightarrow {}^3\mathrm{H} + {}^1\mathrm{H} + \mathrm{Q}. \tag{5.3}$$

The results are two charged ions, which, after amplification using a spark gas(Ar), can be detected as a current.

5.2 The differential cross-section for neutron scattering

Now we would like to find an expression for the measured intensity of scattered neutrons. The calculations in this chapter follows ref. [17] closely, specially chapters 2, 3, 10 and 11. The intensity of the measured neutrons will be described as a differential cross-section pr. angle, since we are interested in having an angular resolution in the result. The measured intensity will be represented by the number of neutrons scattered into the direction of the detector, and most of the time the detector will also be limited to some range in energy. Therefore the basic structure of the cross-section will look like this:

$$\begin{aligned}\frac{d^2\sigma}{d\Omega\, dE_f} = {} & \#\text{ of neutrons scattered per second into } d\Omega \\ & \frac{\text{with final energy between } E_f \text{ and } E_f + dE_f}{\Psi\, d\Omega\, dE_f}.\end{aligned} \tag{5.4}$$

To illustrate the geometry of neutron scattering, see fig. **??**. Let $\vec{r}_j$ be the position of the atom j, and $\vec{k}_i(\vec{k}_f)$ are the vectors of the incoming(outgoing) neutrons. dA is the area of the detector.

Since we are describing inelastic scattering, the energy dependence of the scattering neutrons must be taken specifically into account. The scattered neutrons is a quantum mechanical system, so to find and expression for the number of neutrons scattered in each direction, first we need to define the initial and final states. For this we use complex plane waves:

$$|\psi_i\rangle = \frac{1}{\sqrt{Y}} \exp(i\vec{k}_i \cdot \vec{r}) \tag{5.5}$$

$$|\psi_f\rangle = \frac{1}{\sqrt{Y}} \exp(i\vec{k}_f \cdot \vec{r}). \tag{5.6}$$

Here $\vec{k}_i(\vec{k}_f)$ is the wave-vector of the initial(final) neutron, and $Y = L^3$ appears because the so called box-normalization has been used, where it is assumed that the neutron state is enclosed in a cube of side-length L. This means that

the only allowed states are the ones that have their de Broglie wavelength periodic in the box.

The neutron wave has the velocity $v = \hbar k_i/m_n$, which gives us a resulting incoming neutron flux of:

$$\Psi = |\psi|^2 v = \frac{1}{Y}\frac{\hbar k_i}{m_n}. \tag{5.7}$$

Having defined the states, we can begin to express the number of scattered neutrons into our detector. Quantum mechanically, this relates to the rate of change between the system before and after a neutron has collided with the sample. The answer to this question is called Fermi's Golden Rule. It describes exactly what is needed in our expression, namely the rate of change between initial and final states in a quantum mechanical system of scattering particles. When a quantum mechanical system changes from one initial state $|\psi_i\rangle$ to a final state $|\psi_f\rangle$ where $|\psi_f\rangle$ is part of a continuum of final states. The rate is given as:

$$\mathcal{W}_{|\psi\rangle_i \to |\psi\rangle_f} = \frac{2\pi}{\hbar}\frac{dn}{dE_f}|\langle\lambda_i\psi_i|V|\lambda_f\psi_f\rangle|^2. \tag{5.8}$$

Here the system changes states from λ_i to λ_f and the neutron changes from state ψ_i to ψ_f(to be precise, from σ_i,$\vec{k_i}$ to σ_f,$\vec{k_f}$). Fermi's Golden Rule is the first step, but before we continue the density of states $\frac{dn}{dE}$ has to be determined.

$$\begin{aligned} \frac{dn}{dE_f} &= \frac{dn}{dV_k}\frac{dV_k}{dk_f}\left(\frac{dE_f}{dk_f}\right)^{-1} = \frac{Y}{(2\pi)^3}4\pi k_f^2\frac{m_n}{k_f\hbar^2} = \frac{Y k_f m_n}{2\pi^2\hbar^2} \\ &\downarrow \\ \frac{dn}{dE_f}|_{d\Omega} &= \frac{Y k_f m_n}{(2\pi)^3\hbar^2}d\Omega. \end{aligned} \tag{5.9}$$

For the last step we introduced again the angular dependence, so that now the density of states describes the density of states into a specific direction $\vec{k_f}$ corresponding to a solid angle $d\Omega$.

We are now ready to write the general result for the differential cross section for inelastic neutron scattering. Inserting eq. (5.8) in eq. (5.4) and exchanging the expression for the density of states from eq.(5.9):

$$\frac{d^2\sigma}{d\Omega\, dE_f} = \frac{k_f}{k_i}\left(\frac{m_n}{2\pi\hbar^2}\right)^2|\langle\lambda_i\psi_i|V|\lambda_f\psi_f\rangle|^2\delta(E_{\lambda_i} - E_{\lambda_f} + \hbar\omega). \tag{5.10}$$

The delta-function is introduced to impose energy conservation. If the system has an initial energy $E_{\lambda_i}/E_{\lambda_f}$ and the neutron has an initial/final energy E_i/E_f, we require[18]:

$$E_i + E_{\lambda_i} = E_f + E_{\lambda_f}. \tag{5.11}$$

Now we define ω so that:

$$E_i - E_f = \hbar\omega, \tag{5.12}$$

which gives us the above delta-function.

5.3 Neutron scattering on magnetic samples

To describe the way neutrons interact with magnetic systems in general, we need an expression of the interaction between a neutron and a magnetic particle, usually one or more unpaired electrons of an ion. This is given by the nuclear Zeeman effect:

$$H_Z = -\vec{\mu}_n \cdot \vec{B} = -\gamma\, \mu_N\, \hat{\sigma} \cdot \vec{B}, \tag{5.13}$$

and $\gamma = 1,913$, and

$$\mu_N = \frac{e\hbar}{2m_p}. \tag{5.14}$$

Here $\hat{\sigma}$ are the Pauli matrices. In this case we are describing the effect of the magnetic field created by the electrons of the atom. A good approximation to this is the field of a dipole:

$$\vec{B} = \frac{\mu_0}{4\pi} \nabla \times \left(\vec{\mu} \times \frac{\vec{r}}{r^3} \right), \tag{5.15}$$

where $\vec{\mu}$ is the electron-spin magnetic moment for an electron spin at $\vec{r}_j$:

$$\vec{\mu}_e = -g\, \mu_B\, \vec{s}_j, \tag{5.16}$$

where $\vec{s}_j$ is the spin-operator of the electron, g is called the g-factor and μ_B is the Bohr magneton.

$$\mu_B = \frac{e\hbar}{2m_e}. \tag{5.17}$$

For completeness the relation between the spin-operator and the Pauli matrices are:

$$\vec{s}_i = \frac{\hbar}{2}\sigma_i. \tag{5.18}$$

This we can now insert in eq. (5.13) to get:

$$H_Z = \frac{\vec{\mu}_0}{4\pi}\, g\, \mu_B\, \gamma\, \mu_N\, \hat{\sigma} \cdot \nabla \times \left(\frac{\vec{s} \times (\vec{r} - \vec{r}_j)}{|\vec{r} - \vec{r}_j|^3} \right). \tag{5.19}$$

Summing over all the spin sites, we can use this as the interaction potential $\hat{V}$ in the master equation for neutron scattering eq. (5.10) to get a general expression for the way neutrons interact with magnetic systems:

$$\begin{aligned}\frac{d^2\sigma}{d\Omega\, dE_f} =& \frac{k_i}{k_f}\left(\frac{\mu_0}{4\pi}\right)^2\left(\frac{m_N}{2\pi\hbar^2}\right)^2(\, g\,\mu_B\,\gamma\,\mu_N)^2 \sum_{\lambda_i,\lambda_f,\sigma_i,\sigma_f} p_{\lambda_i} p_{\sigma_i} \\ &\times \left|\left\langle \vec{k}_f\lambda_f\sigma_f \left| \sum_j \hat{\sigma}\cdot\nabla\times\left(\frac{\vec{s}\times(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3}\right)\right| \vec{k}_i\lambda_i\sigma_i\right\rangle\right|^2 \\ &\times \delta(\hbar\omega + E_{\lambda_i} - E_{\lambda_f}),\end{aligned} \tag{5.20}$$

where we have a sum over the Boltzman factors of the initial states:

$$p_{\lambda_i} = \frac{1}{Z} exp(-E_i/k_BT), \tag{5.21}$$

and p_{σ_i} is the spin distribution of the neutrons.

In the general equation there are still some unknown quantities that we need to examine further to find a useful solution. The first one that will be described here is the matrix element:

$$\left\langle \vec{k}_f\lambda_f\sigma_f \left| \sum_j \hat{\sigma}\cdot\nabla\times\left(\frac{\vec{s}\times(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3}\right)\right| \vec{k}_i\lambda_i\sigma_i\right\rangle. \tag{5.22}$$

We see that this matrix element has two parts, one spin-dependent and one space-dependent. The spin-dependent part can be rewritten through a known mathematical identity relating nabla of the cross-product of two vectors to an integral over the same two vectors times an exponential[19]. Then the spin-dependent part is changed to:

$$\begin{aligned}&\left\langle \vec{k}_f\lambda_f\sigma_f \left| \sum_j \hat{\sigma}\cdot\nabla\times\left(\frac{\vec{s}\times(\vec{r}-\vec{r}_j)}{|\vec{r}-\vec{r}_j|^3}\right)\right| \vec{k}_i\lambda_i\sigma_i\right\rangle \\ &\qquad = 4\pi\left\langle \vec{k}_f\lambda_f\sigma_f \left| \sum_j \exp(i\vec{q}\cdot\vec{r}_j)\vec{\sigma}\cdot(\hat{q}\times(\vec{s}\times\hat{q}))\right| \vec{k}_i\lambda_i\sigma_i\right\rangle.\end{aligned}$$

The term:

$$\hat{q}\times(\vec{s}\times\hat{q}) \equiv \vec{s}_\perp, \tag{5.23}$$

is the component of spin perpendicular to $\vec{q}$ which is the observed momentum transfer of the neutrons

$$\vec{q} = \vec{k}_i - \vec{k}_f. \tag{5.24}$$

Therefore we see that only spins perpendicular to $\vec{q}$ are visible to the neutrons. This means that the spin-part can be factorized into a part only dependent on the spin coordinate σ and a part only dependent on the sample coordinate λ. Since we would like a result which does not depend on the final state, we perform a thermal average over the final states:

$$\sum_{\sigma_i,\sigma_f} p_{\sigma_i} |\langle \lambda_f \sigma_f \left| \hat{\sigma} \cdot \vec{s}_\perp \right| \lambda_f \sigma_f \rangle|^2 \tag{5.25}$$
$$= \sum_{\alpha,\beta,\sigma_i} p_{\sigma_i} \langle \sigma_i \left| \sigma^\alpha \sigma^\beta \right| \sigma_i \rangle \langle \lambda_i \left| s_\perp^\beta \right| \lambda_f \rangle \langle \lambda_f \left| s_\perp^\alpha \right| \lambda_f \rangle.$$

Using the assumption that the neutrons are non-polarized, $p_\uparrow = p_\downarrow$ we do the sums over σ_i and σ_f, and finally summing over the final state λ_f, we get:

$$\sum_{\sigma_i,\sigma_f,\lambda_f} p_{\sigma_i} |\langle \lambda_f \sigma_f \left| \hat{\sigma} \cdot \vec{s}_\perp \right| \lambda_f \sigma_f \rangle|^2 \tag{5.26}$$
$$= \sum_\alpha \langle \lambda_i \left| \vec{s}_\perp \cdot \vec{s}_\perp \right| \lambda_i \rangle.$$

For simplicity we redefine the constants:

$$\mu_0 \mu_N \mu_B g \frac{m_N}{2\pi\hbar^2} \gamma = \frac{e^2}{m_e} \frac{\mu_0}{4\pi} \gamma = r_0 \gamma, \tag{5.27}$$

where $r_0 = 2,82\text{fm}$ and is called the electron classical radius. Introducing the fourier transforms of the spin operators:

$$\vec{Q} = \sum_j \exp(i\vec{q} \cdot \vec{r}_j) \vec{s}_j, \tag{5.28}$$

the cross-section now has the form:

$$\frac{d^2\sigma}{d\Omega\, dE_f} = (\gamma r_0)^2 \frac{k_i}{k_f} \sum_{\alpha,\beta} (\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta) \tag{5.29}$$
$$\times \sum_{\lambda_i \lambda_f} p_{\lambda_i} \langle \lambda_i | Q_\alpha | \lambda_f \rangle \langle \lambda_f | Q_\beta | \lambda_i \rangle \delta(\hbar\omega + E_{\lambda_i} - E_{\lambda_f}).$$

The last step is to make another assumption - that the electrons are all located at the positions of the atoms. This means that the position-vector for the electrons and the atoms are equal, except a small deviation equal to the distance of the electron from the nucleus:

$$\vec{Q} = \sum_j \int \exp(i\vec{q} \cdot (\vec{r}_j + \vec{r})) \vec{s}_j d^3\vec{r} = \sum_j \exp(i\vec{q} \cdot \vec{r}_j) \vec{s}_j F(\vec{q}), \tag{5.30}$$

where the magnetic form factor is introduced, given by:

$$F(\vec{q}) = \int \exp(i\vec{q}\cdot\vec{r})d^3\vec{r}. \tag{5.31}$$

Inserting this last result into eq. (5.29), we arrive at the final result:

$$\begin{aligned}\frac{d^2\sigma}{d\Omega\, dE_f} =&(\gamma r_0)^2\frac{k_i}{k_f}\left[\frac{g}{2}F(q)\right]^2\sum_{\alpha,\beta}(\delta_{\alpha\beta}-\hat{q}_\alpha\hat{q}_\beta) \\ &\times\sum_{\lambda_i,\lambda_f}p_{\lambda_i}\sum_{j,j'}\langle\lambda_i|\exp(-i\vec{q}\cdot\vec{r_j})\vec{s}_j^{\,\alpha}|\lambda_f\rangle\langle\lambda_f|\exp(i\vec{q}\cdot\vec{r_{j'}})\vec{s}_{j'}^{\,\beta}|\lambda_i\rangle \\ &\times\delta(\hbar\omega+E_{\lambda_i}-E_{\lambda_f}).\end{aligned} \tag{5.32}$$

This is the master equation for magnetic neutron scattering, and represents the basis for our further calculations. Here we are not taking molecular movements ie. phonons, into account, which would require an extra term $\exp(-2W)$.

We now re-express the delta-function:

$$\delta(\hbar\omega+E_{\lambda_i}-E_{\lambda_f}) = \frac{1}{2\pi\hbar}\int_{-\infty}^{-\infty}dt\exp(-i(\hbar\omega+E_{\lambda_i}-E_{\lambda_f})t/\hbar). \tag{5.33}$$

Inserting this rewriting, the spin-operators are changed to time-dependent Heisenberg operators and the magnetic master equation now takes the form:

$$\begin{aligned}\frac{d^2\sigma}{d\Omega\, dE_f} =&(\gamma r_0)^2\frac{k_i}{k_f}\left[\frac{g}{2}F(q)\right]^2\exp(-2W)\sum_{\alpha,\beta}(\delta_{\alpha\beta}-\hat{q}_\alpha\hat{q}_\beta) \\ &\times N\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dt\exp(-i\omega t)\sum_j\exp(i\vec{q}\cdot\vec{r}_j)\langle\vec{s}_0^{\,\alpha}(0)\vec{s}_j^{\,\beta}(t)\rangle.\end{aligned} \tag{5.34}$$

With this we have arrived at a new form of the previous result, which is useful if we know the Heisenberg operators of the system.

5.3.1 The dynamical correlation function

At any of the 4 previous steps, it is useful to introduce the dynamical correlation function or the dynamical structure factor:

$$\begin{aligned}S^{\alpha\beta}(\vec{q},\omega) =&\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dt\exp(-i\omega t)\sum_j\exp(i\vec{q}\cdot\vec{r}_j)\langle\vec{s}_0^{\,\alpha}(0)\vec{s}_j^{\,\beta}(t)\rangle \\ =&\sum_{j,j'}\langle\lambda_i|\exp(-i\vec{q}\cdot\vec{r_j})\vec{s}_j^{\,\alpha}|\lambda_f\rangle\langle\lambda_f|\exp(i\vec{q}\cdot\vec{r_{j'}})\vec{s}_{j'}^{\,\beta}|\lambda_i\rangle \\ &\times\delta(\hbar\omega+E_{\lambda_i}-E_{\lambda_f}).\end{aligned} \tag{5.35}$$

Depending on whether time-dependent operators are preferred or not. Since there exists many different magnetic systems, there are many varying further treatments of the problem, but we will be focusing on the 1D ferromagnetic system, since that is the solutions found for the anomalous dimensions in $\mathcal{N} = 4$ SYM.

5.4 Determining the dynamic correlation function for a 1D Heisenberg chain with perturbations

To be able to use neutron scattering to tell us something about the Heisenberg chain, we need to determine possible observable quantities of the chains. As we found in the previous section, these are the so called dynamical structure factors, $S^{\alpha\beta}(\vec{q},\omega)$. They are chosen because their Fourier transforms relate (to first order) to the differential magnetic cross section of inelastic neutron scattering described in the previous sections. Much effort has already been put into this, both in determining the dynamical structure factors and in doing measurements that can be used to compare with theory. Stone et al.[20] did a large collaborative work where detailed theoretical predictions were made and compared with neutron scattering measurements. For the Heisenberg XXZ chain they have been calculated, and found to agree very well with experiment[2]. This calculation and measurement is seen in fig. 5.3. There they have used the real-space expression for the dynamical structure factor:

$$S^{\alpha\beta}(m,t) = \frac{\mathrm{Tr}(\sigma_1^\alpha e^{iHt}\sigma_{m+1}^\beta e^{-iHt}e^{\frac{-H}{kT}})}{\mathrm{Tr}(e^{\frac{-H}{kT}})}, \tag{5.36}$$

where m is the lattice distance and t is the time difference.

Unfortunately, that study is not interesting in this case, since only XXX chains appear in the CFT. The Hamiltonian for the XXZ chain includes a preference for the directions of the spins:

$$H_{XXZ} = \sum(1 - \sigma_x \cdot \sigma_x - \sigma_y \cdot \sigma_y - \alpha\sigma_z \cdot \sigma_z). \tag{5.37}$$

Apart from the difference in the , we are not working with the usual Heisenberg Hamiltonian, but with a Hamiltonian with an added perturbation and we know from CFT that this perturbation has the effect of splitting the spin chains.

5.5 Introducing new eigenstates

Therefore the basic solution is not enough. We need to know where the change in the Hamiltonian will have an effect. The equation (5.37) is a very general

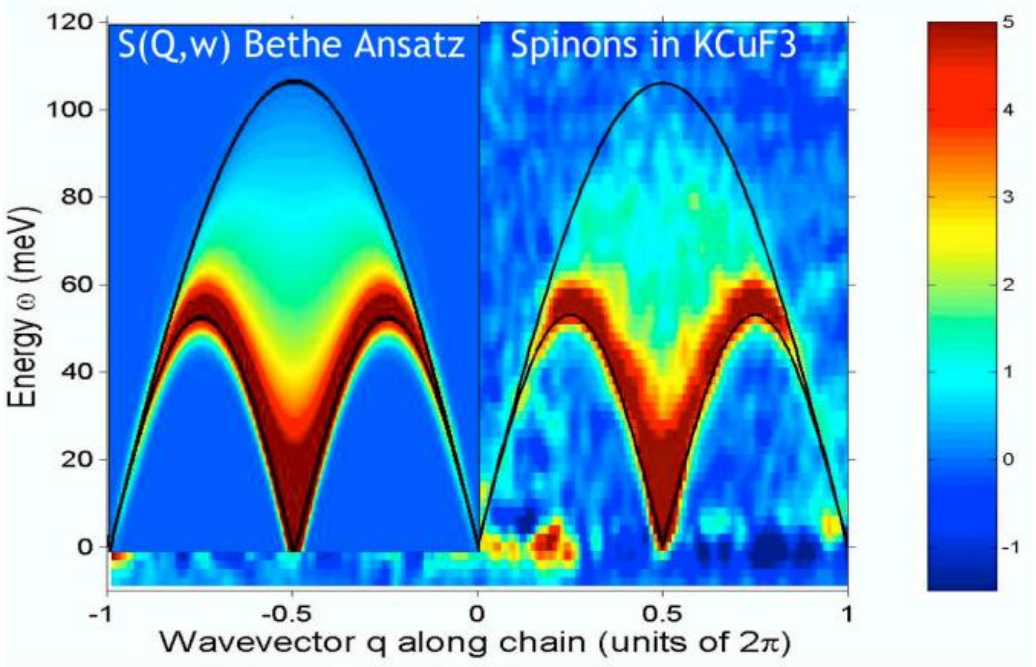


Figure 5.3: Here is shown a comparison between numerical calculations done using the Bethe ansatz and measurements on a spin chain of type XXZ. Figure from ref. [?].

expression for the differential cross section for neutrons scattered on magnetic targets. The specific system that the neutrons are scattered upon comes into play in the initial (λ_i) and final (λ_f) eigenstates from the inner product:

$$\langle \vec{s}_0^{\alpha}(0)\vec{s}_j^{\beta}(t)\rangle = \langle \vec{k}_i, \lambda_i \vec{s}_0^{\alpha}(0)\vec{s}_j^{\beta}(t)|\vec{k}_f, \lambda_f\rangle. \tag{5.38}$$

The final state is summed out since this is the state of the system after the neutrons have interacted with it, and we therefore have no way of measuring it. Therefore, to determine the specific form of $S^{\alpha\beta}(\vec{q},\omega)$, we need to determine the effect of the spin operators $\vec{s}$ on the eigenstates of the new Hamiltonian of the perturbed spin chain system $H = -2 : tr[Z,X][\hat{Z},\hat{X}] :$. To do this we need to relate the operators of the conformal field theory X and Z with the usual spin-operators s_z and $s_\pm$

$$\vec{s}_j^{\beta}|\vec{k}_f, \lambda_f\rangle = \vec{s}_j^{\beta} \sum_{l_1,l_2=1}^{L} \Psi(l_1,l_2)|\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow\downarrow\uparrow\rangle. \tag{5.39}$$

First we notice that the perturbed Hamiltonian does not have any consideration for time-dependence. Therefore it is more logical to use another way of

writing the dynamic correlation function:[21]

$$S^{zz}(\vec{q},\omega) = 2\pi \sum_e |\langle e|s_q^z|0\rangle|^2 \delta(\omega + E_0 - E_e), \tag{5.40}$$

where $\langle e|(|0\rangle)$ are the excited(ground) states, $E_0(E_e)$ are the energies and the expression is valid at $T = 0$ only. To determine this quantity, we need to know how the spin operator $\vec{s}$ works on the states of the system. The problem we are facing is that the states no longer behave like the normal spin states. Some of the eigenstates of the extended dilatation operator will have two or more traces in the states. Which translates, in the spin chain picture, to two or more chains in the state. Therefore we need to determine how the spin-operator will work on these new kind of states.

And educated guess could be that the new state behaves as a kind of superposition, which would mean that applying the spin-operator to the state would produce a sum over two new states - one where the spin operator is used on the first trace, and one where it is used on the other:

$$\begin{aligned} \vec{s}\mathcal{O}_n^{J_1,J_2} &= \vec{s}(\mathrm{Tr}(Z^{J_1}\phi Z\phi)\mathrm{Tr}(Z^{J_2})) \\ &= \vec{s}(\sum_{p=0}^{J} |\uparrow\downarrow\uparrow\downarrow\uparrow\rangle \cdot |\uparrow\uparrow\downarrow\downarrow\rangle), \end{aligned} \tag{5.41}$$

Unfortunately, during this work there was not time to make rigorous tests of this prediction, and that is left as future work. There could also be other ways to use the appearance of spin chains as the planar limit of $N = 4$ SYM. Questions to be asked are:

- What are the most possible states after division/interaction?
- What are the eigenstates after the interaction?

Chapter 6

Results

The previous chapters have shown that a spin chain represents the planar operators in $\mathcal{N} = 4$ SYM, and we also found that it is possible to do measurements on chains of magnetic particles using neutron scattering. Specifically, we found that to determine the form of neutron scattering on the system, we need to investigate the effect of applying the spin operator $\vec{s}$ to the eigenstates of the system. Therefore it is useful to make a specific calculation of the eigenstates of a useful length of chain - in this case length 8 with two impurities, since that length is realizable as one-dimensional magnets.

As we found in part 3.2 the complete 1-loop dilatation operator (including non-planar diagrams) in the scalar SU(N) sector of $\mathcal{N} = 4$ SYM is:

$$\hat{D}_2 = -2 : \mathrm{Tr}([Z, \psi][\hat{Z}, \hat{\psi}]) : \tag{6.1}$$

which means that the expression for the total dilatation-operator is:

$$\hat{D} = \hat{D}_0 + \lambda(\hat{D}_2). \tag{6.2}$$

This full dilatation operator also has non-planar results, which change the number of traces in the operators it works on. In chapter 4 we did not take these into account, since we had learned that the contribution from planar diagrams are N^2-fold larger than the contributions from non-planar diagrams. We found that the planar part equals the Hamiltonian of the Heisenberg XXX spin chain for spin=$\frac{1}{2}$, and its solutions are given by the eigenvalues and eigenstates of that Hamiltonian. This inspires us to make the following re-definition of the dilatation function:

$$D = \hat{D}_0 + \lambda(\hat{D}_2^{\mathrm{planar}} + \frac{1}{N^2}\hat{D}_2^{\mathrm{non-planar}}). \tag{6.3}$$

Now it is clear that if $\hat{D}_2^{\mathrm{non-planar}}$ is treated as a perturbation in quantum mechanical perturbation theory, the eigenstates and eigenvalues of the whole 1-loop dilatation-operator $\hat{D}_2$ can be determined. The only problem is that

the specific expression for $\hat{D}_2^{\text{non-planar}}$ is unknown. So the determination is done by letting the full dilatation operator work on the ground states of the planar dilatation-operator. This gives a result including both single-trace and double trace states. Then the perturbation are represented by the double-trace results, and then quantum mechanical perturbation theory(see appendix B) gives the perturbed eigenstates:

$$\begin{aligned}\psi_n =&\phi_n + \frac{1}{N}\sum_{k\neq n}\frac{\langle\phi_k|H_1|\phi_n\rangle}{E_n^0 - E_k^0} \rightarrow \\ \mathcal{U}_n^J =&\mathcal{O}_n^J + \frac{1}{N}\sum_{m\neq n}\frac{\mathcal{O}_m^J H_1 \mathcal{O}_n^J}{E_n^0 - E_m^0}\mathcal{O}_n^J\end{aligned},$$

and the eigenvalues:

$$\begin{aligned}E_n =&E_n^0 + \frac{1}{N^2}\langle\phi_n|H_1|\phi_n\rangle + \frac{1}{N^2}\langle\phi_n|H_1|\phi_n\rangle \rightarrow \\ E_n =&E_n^0 + \frac{1}{N^2}\mathcal{O}_m^J H_1 \mathcal{O}_n^J + \frac{1}{N^2}\mathcal{O}_m^J H_1 \mathcal{O}_n^J\end{aligned}.$$

To be able to determine these quantities we see that we need to know the result when the dilatation operator is applied to the states $\mathcal{O}_n^J$. This is what will be found in the next section.

6.1 An operator of length 8 with two impurities

The reason that this number of impurities has been chosen is that it appears to be easier to realize this configuration experimentally **Note: find reference from chemistry** To get an overview of the possible operators in the result, we write up all the possible configurations of the fields ϕ and $\mathcal{Z}$. We use BMN operators which means that we have two impurities - 2 ϕ's in a background of 6 $\mathcal{Z}$'s:

$$\mathcal{O}_n^J = \frac{1}{J+1}\sum_{p=0}^{J}\cos\left(\frac{\pi n}{J+1}(2p+1)\right)\mathcal{O}_p^J. \tag{6.4}$$

First the solutions for the planar case are found. There are 6 planar superposition-functions $\mathcal{O}_p^J$, but some of them are equal to each other:

$$\begin{aligned}\mathcal{O}_1^6 = \mathcal{O}_5^6 = Tr(\mathcal{Z}^5\phi\mathcal{Z}\phi) \qquad \mathcal{O}_0^6 = Tr(\mathcal{Z}^6\phi^2) \\ \mathcal{O}_2^6 = \mathcal{O}_4^6 = Tr(\mathcal{Z}^4\phi\mathcal{Z}^2\phi) \qquad \mathcal{O}_3^6 = Tr(\mathcal{Z}^3\phi\mathcal{Z}^3\phi)\end{aligned}. \tag{6.5}$$

This gives us the eigenstate eq. (4.54):

$$\begin{aligned}\mathcal{O}_n^6 &= \frac{1}{7}\sum_{p=0}^{6}\cos\left(\frac{\pi n}{7}(2p-1)\right)\mathcal{O}_p^6 \\ &= \frac{1}{7}\left(\cos\left(\frac{\pi n}{7}(2-1)\right)+\cos\left(\frac{\pi n}{7}(9)\right)\right)Tr(\mathcal{Z}^5\phi\mathcal{Z}\phi) \\ &\quad+\frac{1}{7}\left(\cos\left(\frac{\pi n}{7}(3)\right)+\cos\left(\frac{\pi n}{7}(7)\right)\right)Tr(\mathcal{Z}^4\phi\mathcal{Z}^2\phi) \\ &\quad+\frac{1}{7}\cos\left(\frac{\pi n}{7}(-1)\right)Tr(\mathcal{Z}^5\phi^2)+\frac{1}{7}\cos\left(\frac{\pi n}{7}(5)\right)Tr(\mathcal{Z}^3\phi\mathcal{Z}^3\phi), \qquad (6.6)\end{aligned}$$

where $0 \leq n \leq \frac{J}{2}$. And the eigenvalues are given by the expression (4.36), for $L = 8$:

$$\begin{aligned}E_n &= 4\left[\sin^2\left(\frac{2}{7}\pi n\right)+\sin^2\left(\frac{2}{7}\pi n\right)\right] \qquad (6.7)\\ &= 4\left[1-\cos\left(\frac{2}{7}\pi n\right)\right],\end{aligned}$$

which numerically results in this table: With this result we are ready to go to

n	E/λ
0	0
1	1,5
2	4,89
3	7,6

the non-planar part. Here we have the following possible operators:

$$\mathcal{O}_A = Tr(\mathcal{Z}^5)Tr(\mathcal{Z}\phi^2) \quad \mathcal{O}_B = Tr(\mathcal{Z}^5)Tr(\mathcal{Z}^3\phi^2) \quad \mathcal{O}_C = Tr(\mathcal{Z}^2)Tr(\mathcal{Z}\phi\mathcal{Z}\phi)$$

.

$$\mathcal{O}_D = Tr(\mathcal{Z}^3)Tr(\mathcal{Z}^2\phi^2) \qquad \mathcal{O}_E = Tr(\mathcal{Z}^3)Tr(\mathcal{Z}\phi\mathcal{Z}\phi). \qquad (6.8)$$

Here we take into account that we cannot have the trace of just a single operator - this always gives 0, and that a mixed trace with only one of each operator also does not contribute since it does not mix. We see this by representing the dilatation operator in the following way[22]:

$$(g^2D_2+g^4D_4)\cdot\begin{pmatrix}\mathcal{O}\\ \mathcal{Q}\end{pmatrix} = \begin{pmatrix}* & 0\\ * & 0\end{pmatrix}\cdot\begin{pmatrix}\mathcal{O}\\ \mathcal{Q}\end{pmatrix}. \qquad (6.9)$$

where D is the dilatation operator and the following definitions are used:

$$\mathcal{O}_p^{J_0;J_1,\ldots,J_k} = \mathrm{Tr}(\phi Z^p \phi Z^{J_0-p}) \prod_{i=1}^{k} \mathrm{Tr} Z^{J_i} \tag{6.10}$$

$$\mathcal{Q}^{J_0,J_1;J_2,\ldots,J_k} = \mathrm{Tr}(\phi Z^{J_0})\mathrm{Tr}(\phi Z^{J_1}) \prod_{i=2}^{k} \mathrm{Tr} Z^{J_i}. \tag{6.11}$$

The stars represent various trace-operators that result from working with the dilatationoperator on the $\mathcal{O}$'s and $\mathcal{Q}$'s.

6.2 The planar part of the dilatation operator

The planar part of the dilatation operator has the following representation in the room extended by the symmetric two-impurity states $\mathcal{O}_p^{J_0}$[22], for J=6:

$$D|_{planar} = 4 \cdot \begin{pmatrix} +1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & +2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & +2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & +2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & +2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & +2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & +2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & +1 \end{pmatrix} \tag{6.12}$$

We will now verify this result by doing the explicit calculation.

Since there are 4 independent states eq. (6.5), this is in reality 4 calculations, but only one is shown here because they are very similar. In chapter 3.2, we found that the one-loop dilatation operator is given by:

$$D = -2 : \mathrm{Tr}([Z,\psi][\hat{Z},\hat{\psi}]) := -2(Z_{ab}\psi_{bc} - \psi_{ab}Z_{bc})(\hat{Z}_{cd}\hat{\psi}_{da} - \hat{\psi}_{cd}\hat{Z}_{da}). \tag{6.13}$$

The fields Z and ϕ are fields in SU(N), which means that they are $N \times N$ matrices, and follow the contraction rules in appendix A. In the following calculation the indices are written explicitly to clarify the different steps in the calculation. From appendix A we also know that $\hat{Z}_{cd}$ and $\hat{\psi}_{da}$ can be expressed as the differential operators $\frac{\partial}{\partial Z_{cd}}$ and $\frac{\partial}{\partial \psi_{da}}$, the calculation can start: [ref: appendix C i martins speciale]

$$\begin{aligned} D\mathcal{O}_1^6 = & -2(Z_{ab}\psi_{bc} - \psi_{ab}Z_{bc})\left(\frac{\partial}{\partial Z_{dc}}\frac{\partial}{\partial \psi_{ad}} - \frac{\partial}{\partial \psi_{dc}}\frac{\partial}{\partial Z_{ad}}\right) \\ & \times Z_{12}Z_{23}Z_{34}Z_{45}\psi_{56}Z_{67}\psi_{78}Z_{81} \\ = & (Z_{ab}\psi_{bc} - \psi_{ab}Z_{bc})(\delta_d^5\delta_c^6\psi_{78} + \delta_d^7\delta_c^8\psi_{56}). \end{aligned} \tag{6.14}$$

Here only the $\frac{\partial}{\partial\psi_{dc}}\frac{\partial}{\partial Z_{ad}}$-part of the calculation is useful to do in complete detail, since the other part is done in exactly the same way, just with the indices switched. The numbers $1-8$ are to be understood as representing the place in the trace of the specific Z-field, not as specific Z field. Now, the calculation to be done is this:

$$\begin{aligned}\frac{\partial}{\partial Z_{ad}}\frac{\partial}{\partial\psi_{dc}}(Z_{12}Z_{23}Z_{34}Z_{45}\psi_{56}Z_{67}\psi_{78}Z_{81}) \\ = \frac{\partial}{\partial Z_{ad}}(\delta_d^5\delta_c^6\psi_{78}+\delta_d^7\delta_c^8\psi_{56})(Z_{12}Z_{23}Z_{34}Z_{45}\psi_{56}Z_{67}\psi_{78}Z_{81}),\end{aligned} \tag{6.15}$$

where we see that due to the index expansion, the calculation can be divided into two, one part with the differential operators working on ψ and the other with the differential operators working on Z. Letting the differential operators for the Z's work om the Z-fields gives a total of 6 terms, each with two delta-functions in the indices, which is then multiplied with the two ψ-terms:

$$\begin{aligned}\frac{\partial}{\partial Z_{ad}}\frac{\partial}{\partial\psi_{dc}}(Z_{12}Z_{23}Z_{34}Z_{45}\psi_{56}Z_{67}\psi_{78}Z_{81}) = (\delta_d^5\delta_c^6\psi_{78}+\delta_d^7\delta_c^8\psi_{56}) \\ \times\Big(\delta_a^1\delta_d^2Z_{23}Z_{34}Z_{45}Z_{67}Z_{81}+\delta_a^2\delta_d^3Z_{12}Z_{34}Z_{45}Z_{67}Z_{81} \\ +\delta_a^3\delta_d^4Z_{12}Z_{23}Z_{45}Z_{67}Z_{81}+\delta_a^4\delta_d^5Z_{12}Z_{23}Z_{34}Z_{67}Z_{81} \\ +\delta_a^6\delta_d^7Z_{12}Z_{23}Z_{34}Z_{45}Z_{81}+\delta_a^8\delta_d^1Z_{12}Z_{23}Z_{34}Z_{45}Z_{67}\Big).\end{aligned} \tag{6.16}$$

This gives 12 terms:

$$\begin{aligned}&\delta_a^1\delta_d^2Z_{23}Z_{34}Z_{45}Z_{67}Z_{81}+\delta_a^2\delta_d^3Z_{12}Z_{34}Z_{45}Z_{67}Z_{81}+\delta_a^3\delta_d^4Z_{12}Z_{23}Z_{45}Z_{67}Z_{81} \\ &+\delta_a^4\delta_d^5Z_{12}Z_{23}Z_{34}Z_{67}Z_{81}+\delta_a^6\delta_d^7Z_{12}Z_{23}Z_{34}Z_{45}Z_{81}+\delta_a^8\delta_d^1Z_{12}Z_{23}Z_{34}Z_{45}Z_{67} \\ &+\delta_a^1\delta_d^2Z_{23}Z_{34}Z_{45}Z_{67}Z_{81}+\delta_a^2\delta_d^3Z_{12}Z_{34}Z_{45}Z_{67}Z_{81}+\delta_a^3\delta_d^4Z_{12}Z_{23}Z_{45}Z_{67}Z_{81} \\ &+\delta_a^4\delta_d^5Z_{12}Z_{23}Z_{34}Z_{67}Z_{81}+\delta_a^6\delta_d^7Z_{12}Z_{23}Z_{34}Z_{45}Z_{81}+\delta_a^8\delta_d^1Z_{12}Z_{23}Z_{34}Z_{45}Z_{67}.\end{aligned} \tag{6.17}$$

When now multiplying with $Z_{ab}\psi_{bc}-\psi_{ab}Z_{bc}$ we get back the traces over the fields:

$$\begin{aligned}&-\Big[\mathrm{Tr}(Z^3)\mathrm{Tr}(Z\psi Z^2\psi)+\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^3\psi Z\psi)+\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z\psi Z^4) \\ &+\mathrm{Tr}(\psi Z\psi Z^5)N+\mathrm{Tr}(\psi Z^5)\mathrm{Tr}(Z\psi)+\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\psi Z\psi) \\ &+\mathrm{Tr}(\psi Z^4)\mathrm{Tr}(Z^2\psi)+\mathrm{Tr}(\psi Z^3)\mathrm{Tr}(Z^3\psi)+\mathrm{Tr}(Z^4\psi)\mathrm{Tr}(Z^2\psi) \\ &+\mathrm{Tr}(\psi Z)\mathrm{Tr}(Z^5\psi)+N\mathrm{Tr}(Z^5\psi Z\psi)+\mathrm{Tr}(\psi Z^5)\mathrm{Tr}(Z\psi)\Big] \\ &+\Big[\mathrm{Tr}(Z^3)\mathrm{Tr}(\psi Z\psi Z^2)+\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\psi Z^2\psi)+\mathrm{Tr}(Z)\mathrm{Tr}(\psi ZZ^2\psi Z^3) \\ &+N\mathrm{Tr}(\psi Z^2\psi Z^4)+\mathrm{Tr}(\psi Z^5)\mathrm{Tr}(Z\psi)+\mathrm{Tr}(Z^4)\mathrm{Tr}(\psi^2Z^2) \\ &+\mathrm{Tr}(\psi Z^4)\mathrm{Tr}(\psi Z^2)+\mathrm{Tr}(\psi Z^3)\mathrm{Tr}(Z^3\psi)+\mathrm{Tr}(Z^2\psi)\mathrm{Tr}(\psi Z^4) \\ &+\mathrm{Tr}(\psi Z)\mathrm{Tr}(\psi Z^5)+N\mathrm{Tr}(Z^6\psi^2)+\mathrm{Tr}(Z^5\psi)\mathrm{Tr}(\psi Z)\Big].\end{aligned} \tag{6.18}$$

This gives us half of the expression, the part associated with $\frac{\partial}{\partial Z_{ad}}\frac{\partial}{\partial \psi_{dc}}$. Doing the other part is completely equivalent, and will not be shown here. Adding the two contributions it is found that many of the terms vanish because of the rotational invariance of the traces. This leaves us with:

$$\begin{aligned}\hat{D}\mathcal{O}_1^6 = & -2\Big(2N\mathrm{Tr}(Z^6\psi^2) - 4N\mathrm{Tr}(\psi Z\psi Z^5) + 2N\mathrm{Tr}(Z^4\psi Z^2\psi) \\ & - 2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\psi Z\psi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\psi Z\psi Z^3) - 2\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z\psi Z^4) \\ & + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\psi^2 Z^2) + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\psi Z^2\psi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z^2\psi Z^3)\Big).\end{aligned} \tag{6.19}$$

The results for the three other operators are as follows:

$$\begin{aligned}\hat{D}\mathcal{O}_0^6 = & -2\Big(2N\mathrm{Tr}(\psi^2 Z^6) - 4N\mathrm{Tr}(\psi Z\psi Z^5) + 2N\mathrm{Tr}(Z^4\psi Z^2\psi) \\ & - 2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\psi Z\psi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\psi Z\psi Z^3) \\ & - 2\mathrm{Tr}(Z)\mathrm{Tr}(Z^4\psi Z\psi) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\psi^2 Z^2) \\ & + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\psi Z^2\psi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(Z^3\psi Z^2\psi)\Big).\end{aligned} \tag{6.20}$$

And for the operator $\mathcal{O}_2^6$:

$$\begin{aligned}\hat{D}\mathcal{O}_2^6 = & -\Big(2N\mathrm{Tr}(\psi^2 Z^6) - 4N\mathrm{Tr}(\psi Z\psi Z^5) + 2N\mathrm{Tr}(Z^4\psi Z^2\psi) \\ & - 2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\psi Z\psi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\psi Z\psi Z^3) \\ & - 2\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z\psi Z^4) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\psi^2 Z^2) \\ & + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\psi Z^2\psi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z^2\psi Z^3)\Big).\end{aligned} \tag{6.21}$$

And last, for operator $\mathcal{O}_3^6$:

$$\begin{aligned}\hat{D}\mathcal{O}_3^6 = & -\Big(2N\mathrm{Tr}(\psi^2 Z^6) - 4N\mathrm{Tr}(\psi Z\psi Z^5) + 2N\mathrm{Tr}(Z^4\psi Z^2\psi) \\ & - 2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\psi Z\psi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\psi Z\psi Z^3) \\ & - 2\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z\psi Z^4) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\psi^2 Z^2) \\ & + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\psi Z^2\psi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\psi Z^2\psi Z^3)\Big).\end{aligned} \tag{6.22}$$

As expected, the planar part of the result is N-fold stronger than the non-planar. We also verify the expected form of the planar part of the dilatation operator in the basis of the symmetric two-impurity states $\mathcal{O}_p^J$ to be given by eq. (6.12). This is demonstrated in the fact that all the Q-operators from eq. (6.10) that have an impurity in each trace, disappear. Now it is a simple matter of diagonalizing that matrix in for example Mathematica, to find the

eigenvalues, which are equal to our previous result for the energy(4.36):

$$\begin{aligned} E_n &= 4\left[\sin^2\left(\frac{2}{7}\pi n\right) + \sin^2\left(\frac{2}{7}\pi n\right)\right] \\ &= 4\left[1 - cos\left(\frac{2}{7}\pi n\right)\right]. \end{aligned} \tag{6.23}$$

6.2.1 Changing to basis $\mathcal{O}_n^J$

Now we have the solution in the basis of the symmetric two-impurity states. Now we would like to change into the more physical basis of the real eigenfunctions of the dilatation operator, that we call $\mathcal{O}_n^J$. In other words, we are interested in finding the expression:

$$\begin{aligned} \hat{D}\mathcal{O}_n^J =& \hat{D}\frac{1}{J+1}\sum_{p=0^J}\cos\left(\pi n\frac{2p+1}{J+1}\right)\mathcal{O}_p^J \\ =& \frac{1}{7}\sum_{p=0}^{6}\cos\left(\pi n\frac{2p+1}{7}\right)\hat{D}\mathcal{O}_p^6. \end{aligned} \tag{6.24}$$

Fortunately, we have already calculated the expressions $\hat{D}\mathcal{O}_p^6$. For each of the $\mathcal{O}_p^6$'s we got both operators with higher and lower p, as well as single and double trace operators. The single-trace operators represents the planar part, and as a test we will start by isolating that part. Inserting the results from eq.s (6.19) to (6.22) in the expression (6.24) and isolating the parts only dependent on single-trace operators, we end up with:

$$\begin{aligned} \hat{D}\mathcal{O}_n^6 =& \mathrm{Tr}(\psi^2 Z^6)[-2(\cos(-\frac{\pi n}{7}) + \cos(-\pi n\frac{11}{7})) + 2(\cos(\frac{\pi n}{7}) + \cos(-\pi n\frac{7}{7}))] \\ &+ \mathrm{Tr}(\psi Z\psi Z^5)[2(\cos(-\frac{\pi n}{7}) + \cos(\pi n\frac{11}{7})) - 4(\cos(\frac{\pi n}{7}) + \cos(\pi n\frac{9}{7})) \\ &+ 2(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{7}{7}))] \\ &+ \mathrm{Tr}(Z^4\psi Z^2\psi)[2(\cos(\frac{\pi n}{7}) + \cos(\pi n\frac{9}{7})) - 4(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{7}{7})) + 4\cos(\pi n)] \\ &+ \mathrm{Tr}(\psi Z^3\psi Z^3)[-4\cos(\pi n) + 2(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{7}{7}))]. \end{aligned} \tag{6.25}$$

This is just the planar part of the solution, so it should simplify to:

$$\hat{D}\mathcal{O}_n^J|_{planar} = E_n\mathcal{O}_n^J = E_n\frac{1}{J+1}\sum_{p=0}^{J}\cos\left(\frac{\pi n(2p+1)}{J+1}\right)\mathcal{O}_p^J, \tag{6.26}$$

where E_n are the eigenvalues found earlier in eq. (6.13). To see this explicitly for this result, we use sine and cosine product and sum relations to rewrite eq. (6.25) to:

$$\begin{aligned}\hat{D}\mathcal{O}^6_n|_{planar} =&\mathrm{Tr}(\psi^2 Z^6)[4\cos(\frac{\pi n}{7})(\cos(\pi n\frac{2}{7})-1)] \\ &+\mathrm{Tr}(\psi Z\psi Z^5)[4\cos(\pi n\frac{2}{7})(\cos(\pi n\frac{2}{7})-1)] \\ &+\mathrm{Tr}(\psi Z^2\psi Z^4)[4\cos(\pi n\frac{5}{7})(\cos(\pi n\frac{2}{7})-1)] \\ &+\mathrm{Tr}(\psi Z^3\psi Z^3)[4\cos(\pi n\frac{9}{7})(\cos(\pi n\frac{2}{7})-1)],\end{aligned} \tag{6.27}$$

Where we see that the two factors in the square brackets correspond to E_n times the cosine factor in the definition of $\mathcal{O}^J_n$ (6.24). To demonstrate the method used in this rewriting, here is an example for the factor in front of $\mathrm{Tr}(\psi^2 Z^6)$:

$$\begin{aligned}&-2(\cos(\pi n\frac{1}{7})+\cos(\pi n\frac{13}{7}))+2(\cos(\pi n\frac{3}{7})+\cos(\pi n\frac{11}{7})) \\ =\;&-2(\cos(\pi n\frac{1}{7})+\cos(\pi n\frac{13}{7}))+2(\cos(\pi n(\frac{1}{7}+\frac{2}{7})+\cos(\pi n(\frac{13}{7}+\frac{-2}{7})) \\ =\;&-2(\cos(\pi n\frac{1}{7})+\cos(\pi n\frac{13}{7})) \\ &+2\cos(\pi n(\frac{1}{7}))\cos(\pi n(\frac{2}{7}))+2\sin(\pi n(\frac{1}{7})\sin(\pi n(\frac{2}{7})) \\ &+2\cos(\pi n(\frac{13}{7}))\cos(\pi n(\frac{-2}{7}))-2\sin(\pi n(\frac{13}{7}))\sin(\pi n(\frac{-2}{7})) \\ =\;&-2(\cos(\pi n\frac{1}{7})+\cos(\pi n\frac{13}{7})) \\ &+2\cos(\pi n(\frac{2}{7}))(\cos(\pi n(\frac{1}{7}))+\cos(\pi n(\frac{13}{7})))+\cos(\pi n(\frac{1}{7})) \\ &-\cos(\pi n(\frac{3}{7}))-\cos(\pi n(\frac{13}{7}))+\cos(\pi n(\frac{11}{7})) \\ =\;&\Big(2\cos(\pi n(\frac{2}{7}))-2\Big)\Big(\cos(\pi n\frac{-1}{7})+\cos(\pi n\frac{11}{7})\Big).\end{aligned} \tag{6.28}$$

Now we have the results for the linear part, next we need to determine the non-linear part of the result. Again inserting the results from eq.s (6.19) to (6.22)

into eq. (6.24), but now isolating all the non-linear terms to get:

$$
\begin{aligned}
&\hat{D}\mathcal{O}_n^J|_{non-planar} \qquad (6.29)\\
&= (\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7})) \times [-2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z\phi Z^3)\\
&\quad - 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z\phi Z^4) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi^2 Z^2)\\
&\quad + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^3\phi Z^2)]\\
&+(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7}))\\
&\quad \times [-2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z^3\phi Z)\\
&\quad - 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^4\phi Z) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi^2 Z^2)\\
&\quad + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^2\phi Z^3)]\\
&+(\cos(\pi n\frac{5}{7}) + \cos(\pi n\frac{9}{7}))\\
&\quad \times [-2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z\phi Z^3)\\
&\quad - 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z\phi Z^4) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi^2 Z^2)\\
&\quad + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^2\phi Z^3)]\\
&+\cos(\pi n\frac{7}{7})\\
&\quad \times [-2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z^3\phi Z)\\
&\quad - 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^4\phi Z) + 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi^2 Z^2)\\
&\quad + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^2\phi Z^3)]\\
&= \mathrm{Tr}(Z^2)[\mathrm{Tr}(Z^4\phi^2)(-\cos(\pi n\frac{1}{7}) - \cos(\pi n\frac{13}{7}) + 2\cos(\pi n\frac{7}{7}))\\
&\quad + \mathrm{Tr}(Z^2\phi Z^2\phi)[2(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})) - 2(\cos(\pi n\frac{5}{7}) + \cos(\pi n\frac{9}{7}))]\\
&\quad + \mathrm{Tr}(Z^3\phi Z\phi)(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7}) - 2(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})) + 2(\cos(\pi n\frac{5}{7}) + \cos(\pi n\frac{9}{7}))\\
&\qquad - 2\cos(\pi n\frac{7}{7}))]\\
&+\mathrm{Tr}(Z^3)[\mathrm{Tr}(Z^3\phi^2)(-(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7})) + 2(\cos(\pi n\frac{5}{7}) + \cos(\pi n\frac{9}{7})))\\
&\quad + \mathrm{Tr}(Z^2\phi Z\phi)(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7}) - 2(\cos(\pi n\frac{5}{7}) + \cos(\pi n\frac{9}{7})))]\\
&+\mathrm{Tr}(Z^4)[\mathrm{Tr}(Z^2\phi^2)(-(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7})) + 4(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})))\\
&\quad + \mathrm{Tr}(Z\phi Z\phi)(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7}) - 4(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})))].
\end{aligned}
$$

This can also be rearranged to fit with the general expression in eq. (6.24), from which we get the full spectrum of the dilatation operator, to first order

in quantum mechanical perturbation theory. To change this into a function of the eigenfunctions, we will use the relation which goes from O_p^J to O_n^J:

$$\mathcal{O}_p^J = \mathcal{O}_{n=0}^J + 2\sum_{n=1}^{[J/2]} \cos\left(\frac{\pi n(2p+1)}{J+1}\right) O_n^J. \tag{6.30}$$

Inserting this expression for O_p^J in eq. (6.29) we get:

$$\begin{aligned}
&\hat{D}\mathcal{O}_n^J|_{non-planar} \qquad (6.31)\\
&= \mathrm{Tr}(Z^2) \times [(\mathcal{O}^4_{n=0} + \cos(\pi n\frac{1}{5})O^4_{n=1} + \cos(\pi n\frac{1}{5})O^4_{n=2})(-\cos(\pi n\frac{1}{7}) - \cos(\pi n\frac{13}{7}) + 2\cos(\pi n\frac{7}{7}\\
&\quad + (\mathcal{O}^4_{n=0} + \cos(\pi n\frac{5}{5})O^4_{n=1} + \cos(\pi n\frac{5}{5})O^4_{n=2})(2(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})) - 2(\cos(\pi n\frac{5}{7}) + \cos\\
&\quad + (\mathcal{O}^4_{n=0} + \cos(\pi n\frac{3}{5})O^4_{n=1} + \cos(\pi n\frac{3}{5})O^4_{n=2})(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7}) - 2(\cos(\pi n\frac{3}{7}) + \cos(\pi\\
&\quad + \mathrm{Tr}(Z^3)\\
&\quad \times [(\mathcal{O}^3_{n=0} + \cos(\pi n\frac{1}{4})O^3_{n=1} + \cos(\pi n\frac{1}{4})O^3_{n=2})(-(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7})) + 2(\cos(\pi n\frac{5}{7}) + c\\
&\quad + (\mathcal{O}^3_{n=0} + \cos(\pi n\frac{3}{4})O^3_{n=1} + \cos(\pi n\frac{3}{4})O^3_{n=2})(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7}) - 2(\cos(\pi n\frac{5}{7}) + \cos(\pi\\
&\quad \mathrm{Tr}(Z^4)\\
&\quad \times [(\mathcal{O}^2_{n=0} + \cos(\pi n\frac{1}{3})O^2_{n=1})(-(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7})) + 4(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})))\\
&\quad + (\mathcal{O}^2_{n=0} + \cos(\pi n\frac{3}{3})O^2_{n=1})(\cos(\pi n\frac{1}{7}) + \cos(\pi n\frac{13}{7}) - 4(\cos(\pi n\frac{3}{7}) + \cos(\pi n\frac{11}{7})))].
\end{aligned}$$

Until this point all calculation have been done to arbitrary n, but to get the specific form of the dilatation operator, it will be too cumbersome to continue in that way. Therefore we now specialize to the three values that n takes, $n = [0, J/2] = 0, 1, 2$, which gives us the results,
first for $n = 0$:

$$\hat{D}\mathcal{O}_0^{n=6}|_{non-planar} = 2.82843\mathcal{O}_0^{n=3}TrZ3 + 9.\mathcal{O}_1^{n=2}TrZ4, \tag{6.32}$$

for $n = 1$:

$$\hat{D}\mathcal{O}^6_{n=1}|_{non-planar} = -6.58901\mathcal{O}^4_{n=1}TrZ2 + 1.8711\mathcal{O}^4_{n=2}TrZ2 - 6.07532\mathcal{O}^3_{n=1}TrZ3 - 0.0326554\mathcal{O} \tag{6.33}$$

and last, for $n = 2$: REACHED 'HERE - STILL NEEDS CHECKING

$$\hat{D}\mathcal{O}^6_2|_{non-planar} = 8.88178 \cdot 10^{-16}\mathcal{O}^4_0\mathrm{Tr}Z^2 + 2.71709\mathcal{O}^4_1\mathrm{Tr}Z^2 - 4.06739\mathcal{O}^4_2\mathrm{Tr}Z^2 - 3.02226\mathcal{O}^3_1\mathrm{Tr}Z^3 \tag{6.34}$$

With this we have the full solution for the planar part of the dilatation operator, and are ready to go on to the non-planar part.

6.3 Double and triple trace operators

Since some of the results of using the dilatation operator on the single-trace operators are double trace operators, we need to know how the dilatation operators works on these operators as well. There are 7 double trace operators:

$$\mathcal{O} = \mathrm{Tr}(Z^2)Tr(Z^4\phi^2),\mathcal{O}_E = \mathrm{Tr}(Z^2)Tr(Z^3\phi Z\phi), \quad \mathcal{O}_F = \mathrm{Tr}(Z^2)Tr(Z^2\phi Z^2\phi),\mathcal{O}_F = \mathrm{Tr}(Z^2)Tr(Z^2\phi Z^2\phi$$

$$\mathcal{O}_G = \mathrm{Tr}(Z^3)Tr(Z^3\phi^2),\mathcal{O}_H = \mathrm{Tr}(Z^3)Tr(Z^2\phi Z\phi), \quad \mathcal{O}_I = \mathrm{Tr}(Z^4)Tr(Z^2\phi^2)\mathcal{O}_J = \mathrm{Tr}(Z^4)Tr(Z\phi Z\phi). \tag{6.35}$$

The calculation follows the same procedure as the one shown in eq. (6.15) and following, the only difference being that we now have two traces on which the differential operators on Z can be applied:

$$\frac{\partial}{\partial Z_{ad}}\frac{\partial}{\partial \psi_{dc}}(\psi_{12}\psi_{23}Z_{34}Z_{45}Z_{51}Z_{1'2'}Z_{2'3'}Z_{3'1'}). \tag{6.36}$$

This means that we will have instances where a contraction happens between the two traces, and this results in the recreation of the original trace of length 8. The results for the planar operators were divided into planar and non-planar, but for the rest of the calculations, we will state both kinds of results together. This way the results look more compressed, which is preferable now that the procedure has been well established through the results for the planar operators.

The results for the double trace operators are:

$$\hat{D}\mathcal{O}_0^4\mathrm{Tr}(Z^2) = 2\mathrm{Tr}(Z^5\phi Z\phi) + 2N\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z\phi Z^3) + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z\phi Z\phi) \tag{6.37}$$
$$+ 4\mathrm{Tr}(Z^5\phi Z\phi) - 4\mathrm{Tr}(Z^4\phi Z^2\phi) - 2N\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi^2 Z^4)$$
$$- 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi^2 Z^2) - 4\mathrm{Tr}(Z^5\phi^2).$$

And for the operator $\mathcal{O}_1^4\mathrm{Tr}(Z^2)$ - TJEKKES:

$$\hat{D}\mathcal{O}_1^4\mathrm{Tr}(Z^2) = 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi^2) + 4N\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) + 2N\mathrm{Tr}(Z^4\phi Z^2\phi) \tag{6.38}$$
$$- 2\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi) - 2\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z\phi Z^3) - 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z\phi Z^4)$$
$$+ 2\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi^2 Z^2) + 2\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) + 2\mathrm{Tr}(Z)\mathrm{Tr}(\phi Z^2\phi Z^3)).$$

And for the operator $\mathcal{O}_2^4\mathrm{Tr}(Z^2)$:

$$\hat{D}\mathcal{O}_2^4\mathrm{Tr}(Z^2) = 2N\mathrm{Tr}(Z^3\phi Z\phi)\mathrm{Tr}(Z^2) + 4\mathrm{Tr}(\phi Z^3\phi Z^3) + 2N\mathrm{Tr}(Z^3\phi Z\phi)\mathrm{Tr}(Z^2) \tag{6.39}$$
$$+ 4\mathrm{Tr}(Z^3\phi Z^3\phi) - 2N\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi) - 4\mathrm{Tr}(Z^4\phi Z^2\phi)$$
$$- 2N\mathrm{Tr}(Z^2)\mathrm{Tr}(\phi Z^2\phi Z^2) - 4\mathrm{Tr}(Z^4\phi Z^2\phi).$$

For operator $\mathcal{O}_0^3\mathrm{Tr}(Z^3)$:

$$\hat{D}\mathcal{O}_0^3\mathrm{Tr}(Z^3) = 2N\mathrm{Tr}(Z^3)\mathrm{Tr}(\phi Z^2\phi Z) + 6\mathrm{Tr}(Z^4\phi Z^2\phi) + 6\mathrm{Tr}(Z^5\phi Z\phi) \quad (6.40)$$
$$- N2\mathrm{Tr}(Z^3)\mathrm{Tr}(Z^3\phi^2) - 6\mathrm{Tr}(Z^6\phi^2) - 6\mathrm{Tr}(Z^3\phi Z^3\phi).$$

$$\hat{D}\mathcal{O}_1^3\mathrm{Tr}(Z^3) = 2N\mathrm{Tr}(Z^3)\mathrm{Tr}(Z^2\phi^2) + 6\mathrm{Tr}(Z^3\phi Z^3\phi) + 6\mathrm{Tr}(Z^4\phi Z^2\phi) \quad (6.41)$$
$$- 2N\mathrm{Tr}(Z^3)\mathrm{Tr}(Z^2\phi Z\phi) - 6\mathrm{Tr}(Z^5\phi Z\phi) - 6\mathrm{Tr}(Z^4\phi Z^2\phi).$$

And last for the operator $\mathcal{O}_0^2\mathrm{Tr}(Z^4)$:

$$\hat{D}\mathcal{O}_0^2\mathrm{Tr}(Z^4) = 2N\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi Z\phi Z) + 8\mathrm{Tr}(Z^3\phi Z^3\phi) + 8\mathrm{Tr}(Z^5\phi Z\phi) \quad (6.42)$$
$$- 2N\mathrm{Tr}(Z^4)\mathrm{Tr}(Z^2\phi^2) - 8\mathrm{Tr}(\phi Z^2\phi Z^4) - 8\mathrm{Tr}(Z^6\phi^2).$$

$$\hat{D}\mathcal{O}_1^2\mathrm{Tr}(Z^4) = +4N\mathrm{Tr}(Z^4)\mathrm{Tr}(Z^2\phi^2) + 16\mathrm{Tr}(Z^4\phi Z^2\phi) - 4N\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi) \quad (6.43)$$
$$- 16\mathrm{Tr}(Z^5\phi Z\phi).$$

Again we need to rewrite so that we get a result expressed in the $\mathcal{O}_n^J$ instead of $\mathcal{O}_n^J$, where again eq. (6.30) is used. We change to numerical calculation since the sums of cosine and sines become too long.

$$\begin{aligned}
&\hat{D}\mathcal{O}_n^4\mathrm{Tr}(Z^2) \quad (6.44)\\
&= \mathrm{Tr}(Z^5\phi Z\phi)[8(\cos(\pi n\tfrac{1}{5}) + \cos(\pi n\tfrac{9}{5})) - 4(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5}))]\\
&\quad + \mathrm{Tr}(Z^4\phi Z^2\phi)[-4(\cos(\pi n\tfrac{1}{5}) + \cos(\pi n\tfrac{9}{5})) + 8(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5})) - 8\cos(\pi n\tfrac{5}{5})]\\
&\quad + \mathrm{Tr}(Z^3\phi Z^3\phi)[-4(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5})) + 8\cos(\pi n\tfrac{5}{5})]\\
&\quad + \mathrm{Tr}(Z^6\phi^2)[-4(\cos(\pi n\tfrac{1}{5}) + \cos(\pi n\tfrac{9}{5}))]\\
&\quad + N\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^3\phi Z\phi)[2(\cos(\pi n\tfrac{1}{5}) + \cos(\pi n\tfrac{9}{5})) - 4(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5})) + 4\cos(\pi n\tfrac{5}{5})]\\
&\quad + N\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi Z^2\phi)[2(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5})) - 4\cos(\pi n\tfrac{5}{5})]\\
&\quad + N\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^4\phi^2)[-2(\cos(\pi n\tfrac{1}{5}) + \cos(\pi n\tfrac{9}{5})) + 2(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5}))]\\
&\quad + \mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z\phi Z\phi)[2(\cos(\pi n\tfrac{1}{5}) + \cos(\pi n\tfrac{9}{5})) - 2(\cos(\pi n\tfrac{3}{5}) + \cos(\pi n\tfrac{7}{5}))].
\end{aligned}$$

As expected, the planar results, given by the results with double trace, are N-fold stronger than the rest. The results for the other non-planar operators

are the following:

$$\begin{aligned}
&\hat{D}\mathcal{O}_n^3\mathrm{Tr}(Z^3) \qquad (6.45)\\
&= \mathrm{Tr}(Z^5\phi Z\phi)[6(\cos(\pi n\frac{1}{4})+\cos(\pi n\frac{7}{4}))-6(\cos(\pi n\frac{3}{4})+\cos(\pi n\frac{5}{4}))]\\
&\quad+\mathrm{Tr}(Z^4\phi Z^2\phi)[6(\cos(\pi n\frac{1}{4})+\cos(\pi n\frac{7}{4}))]\\
&\quad+\mathrm{Tr}(Z^3\phi Z^3\phi)[-6(\cos(\pi n\frac{1}{4})+\cos(\pi n\frac{7}{4}))+6(\cos(\pi n\frac{3}{4})+\cos(\pi n\frac{5}{4}))]\\
&\quad+\mathrm{Tr}(Z^6\phi^2)[-6(\cos(\pi n\frac{1}{4})+\cos(\pi n\frac{7}{4}))]\\
&\quad+N\mathrm{Tr}(Z^3)\mathrm{Tr}(Z^2\phi Z\phi)[2(\cos(\pi n\frac{1}{4})+\cos(\pi n\frac{7}{4}))-(\cos(\pi n\frac{3}{4})+\cos(\pi n\frac{5}{4}))]\\
&\quad+N\mathrm{Tr}(Z^3)\mathrm{Tr}(Z^3\phi^2)[-2(\cos(\pi n\frac{1}{4})+\cos(\pi n\frac{7}{4}))+(\cos(\pi n\frac{3}{4})+\cos(\pi n\frac{5}{4}))],
\end{aligned}$$

$$\begin{aligned}
&\hat{D}\mathcal{O}_n^2\mathrm{Tr}(Z^4) \qquad (6.46)\\
&= \mathrm{Tr}(Z^5\phi Z\phi)[8(\cos(\pi n\frac{1}{3})+\cos(\pi n\frac{5}{3}))-16\cos(\pi n\frac{3}{3})]\\
&\quad+\mathrm{Tr}(Z^4\phi Z^2\phi)[-8(\cos(\pi n\frac{1}{3})+\cos(\pi n\frac{5}{3}))+16\cos(\pi n\frac{3}{3})]\\
&\quad+\mathrm{Tr}(Z^3\phi Z^3\phi)[8(\cos(\pi n\frac{1}{3})+\cos(\pi n\frac{5}{3}))]\\
&\quad+\mathrm{Tr}(Z^6\phi^2)[-8(\cos(\pi n\frac{1}{3})+\cos(\pi n\frac{5}{3}))]\\
&\quad+N\mathrm{Tr}(Z^4)\mathrm{Tr}(Z\phi Z\phi)[2(\cos(\pi n\frac{1}{3})+\cos(\pi n\frac{5}{3}))-4\cos(\pi n\frac{3}{3})]\\
&\quad+N\mathrm{Tr}(Z^4)\mathrm{Tr}(\phi^2Z^2)[-2(\cos(\pi n\frac{1}{3})+\cos(\pi n\frac{5}{3}))+4\cos(\pi n\frac{3}{3})].
\end{aligned}$$

And triple-trace operator results:

$$\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z\phi Z\phi) = . \qquad (6.47)$$

$$\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2)\mathrm{Tr}(Z^2\phi^2) = . \qquad (6.48)$$

Again, all results are re-expressed in the real eigenstates $\mathcal{O}_n^J$ by using eq: (6.30):

$$\hat{D}\mathcal{O}^4_{n=0}\mathcal{O}^2 = -1.08552\mathcal{O}^6_{n=1} - 1.93598\mathcal{O}^6_{n=2} + 0.64155\mathcal{O}^6_{n=3}, \tag{6.49}$$
$$\hat{D}\mathcal{O}^4_{n=1}\mathcal{O}^2 = -1.34164\mathcal{O}^2_{n=1} - 1.38197N\mathcal{O}^4_{n=1} + 1.05145\mathcal{O}^6_{n=1} - 4.13147\mathcal{O}^6_{n=2} + 2.12431\mathcal{O}^6_{n=3}$$
$$\hat{D}\mathcal{O}^4_{n=2}\mathcal{O}^2 = -1.34164\mathcal{O}^2_{n=1} - 3.61803N\mathcal{O}^4_{n=2} + 0.0881656\mathcal{O}^6_{n=1} + 1.7266\mathcal{O}^6_{n=2} - 5.80034\mathcal{O}^6_{n=3}$$
$$\hat{D}\mathcal{O}^3_{n=0}\mathcal{O}^3 = -0.707107N\mathcal{O}^3_{n=1} - 4.57338\mathcal{O}^6_{n=1} - 1.59461\mathcal{O}^6_{n=2} + 2.03534\mathcal{O}^6_{n=3}$$
$$\hat{D}\mathcal{O}^3_{n=1}\mathcal{O}^3 = -1.5N\mathcal{O}^3_{n=1} + 1.95285\mathcal{O}^6_{n=1} - 9.19269\mathcal{O}^6_{n=2} + 8.32709\mathcal{O}^6_{n=3}$$
$$\hat{D}\mathcal{O}^2_{n=0}\mathcal{O}^4 = -4.N\mathcal{O}^2_{n=1} + 1.33333\mathcal{O}^6_{n=0} - 8.93721\mathcal{O}^6_{n=1} + 4.09727\mathcal{O}^6_{n=2} - 6.22342\mathcal{O}^6_{n=3}$$
$$\hat{D}\mathcal{O}^2_{n=1}\mathcal{O}^4 = 1.N\mathcal{O}^2_{n=1} + 0.666667\mathcal{O}^6_{n=0} + 2.29948\mathcal{O}^6_{n=1} - 3.37895\mathcal{O}^6_{n=2} - 5.33154\mathcal{O}^6_{n=3}.$$

With these results, it is now possible to write the full dilatation operator on matrix form, now in the basis of $\mathcal{O}_n^L$:

$$(\mathcal{O}^6_{n=0}, \mathcal{O}^6_{n=1}, \mathcal{O}^6_{n=2}, \mathcal{O}^6_{n=3}, \mathcal{O}^4_{n=0}\mathcal{O}^2, \mathcal{O}^4_{n=1}\mathcal{O}^2, \mathcal{O}^4_{n=2}\mathcal{O}^2, \mathcal{O}^3_{n=0}\mathcal{O}^3, \mathcal{O}^3_{n=1}\mathcal{O}^3, \mathcal{O}^2_{n=0}\mathcal{O}^4, \mathcal{O}^2_{n=1}\mathcal{O}^4, \mathcal{O}^2_{n=0}\mathcal{O} \tag{6.50}$$

which is given by the following matrix:

$$
\hat{D} = -2 \cdot \begin{pmatrix}
0 & 0 & 0 & 0 & -2.00 & 1.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0 & 0 \\
0 & N1.50604 & 0 & 0 & 1.24 & -6.97 & -6.97 & 0.00 & -6.07 & 0.00 & 0.00 & 0.01 & 0 & 0 \\
0 & 0 & N4.89008 & 0 & 0.44 & -4.42 & -4.42 & 0.00 & 0.00 & 0.00 & 0.00 & 12.68 & 0 & 0 \\
0 & 0 & 0 & N7.60388 & -1.80 & 12.30 & 12.30 & 0.00 & 4.46 & 4.46 & 0.00 & 0.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N1.50604 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & N4.89008 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & N1.50604 & 0 & N4.89008 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N1.50604 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

With this the full spectrum of the dilatation operator is determined. As expected, we see that because of the 1-loop diagrams, non-planar diagrams with more than one trace have appeared as part of the Hilbert space of the dilatation operator.

Using this result, it is now possible to calculate the perturbative contribution to the anomalous dimension.

Chapter 7

Conclusions

Relating two fields to each other, when they are each at its own extrema in physics, is not easy. Theorists think anything can be measured - experimentalists think anything can be derived. They each expect immense things from each other, and cannot understand why the other cannot live up to these expectations.

We showed that the planar anomalous dimensions in $\mathcal{N} = 4$ SYM is represented as the eigenvalues of the Heisenberg chain Hamiltonian. In this way a easily found solution of all the planar anomalous dimensions is achieved. This relation is what inspired the main result of this work. Since the planar part of the diagrams are directly related to a quantum-mechanical system, we were led to speculate that also non-planar diagrams might be related to similar quantum mechanical systems.

We therefore determined the eigenvalues and eigenstates of the 1-loop dilatations-operator of SYM, for an operator of length 8. This result is the first step towards determining the structural dynamic correlation functions for the splitting chain. We found that the 1-loop dilatations-operator creates double- and even triple trace states, corresponding to the spin chain picture of a chain being divided into 2 or 3 new chains. Based on this, we are able to conclude that to understand neutron scattering experiments on splitting chains, we need to develop a new expression for the spin-operator. It is not clear how the normal $\vec{s}_i$ would work if applied to a multi-trace state, these states are not even part of the Hilbert space. Even though a straight forward superposition-like approach seems plausible, there was not time in this work to pursue this further. It would be an interesting further development.

Another result is that only states with both impurities in the same trace are eigenstates to the new Hamiltonian. This would hint at some resulting attraction between the impurities, which results in them always ending up in the same chain.

Chapter 8

Outlook

Now the first steps have been taken towards the goal of formulating the way spin chains can relate solid state physics with high energy physics and string theory.

Determination of the new spin-operator.

Both excitations are in the same trace always -¿ Bound states in spin-wave theory

A logical further work inspired by this thesis would be to actually try to measure the magnetic properties of spin chains. In chemistry they are called spin rings when they are rotationally symmetric, and examples exist of rings of particles that behave in 1-dimensional ways[**?**]. In fig. **??** one of these spin rings is shown, and through discussion with Jesper Bendix(NBI), it seems that it is possible to create ferromagnetic rings with two impurities like the ones corresponding to BNM operators. Of course one would have to take into account the 3-dimensional nature of the rings when doing the neutron scattering. This means that the structure factors become 2-dimensional, which was not covered in this work. Also the question of how to create the 'breaking' of the spin rings comes into mind - maybe this could also be related in some way to the bound states treatment in spin-wave theory mentioned before.

One area which was not investigated in this work is the string theory side of the AdS/CFT correspondence. Since the anomalous dimensions of the CFT is directly corresponding to mass (or energy) of string states, and since these anomalous dimensions are described (in the planar limit) by the Heisenberg Hamiltonian, maybe the spin chains can be a model of strings in that limit? Even more fascinating, if the spin chain analogy could be carried to higher loop order, it would mean that interacting strings could be modeled by spin chains breaking(rejoining).

Appendix A

Contraction rules in gauge-groups

To be able to do calculations on the fields of $\mathcal{N} = 4$ SYM, we need to know the contraction rules of the gauge groups that we will be looking at. Since we are not investigating any dependence on space-time, the rules will only cover the combinatorial part of the contraction rules. For simplicity we choose to normalize all the generators with the same constant[15]:

$$\mathrm{Tr}(T^a T^b) = a\delta^{ab}. \tag{A.1}$$

This means that the action is the same for all gauge groups, which is useful if we were taking into account different feynman rules. To find the contraction rules we need the completeness relation for SU(N):

$$(T^a)^\beta_\alpha (T^a)^\mu_\nu) = a(\delta^\beta_\nu \delta^\mu_\alpha - \frac{1}{N}\delta^\beta_\alpha \delta^\mu_\nu). \tag{A.2}$$

This shows us that the contraction $\mathrm{Tr}(Z\hat{Z})$ corresponds to a delta-function in the indices so that:

$$\mathrm{Tr}((T^a)^\beta_\alpha (T^a)^\mu_\nu) = \delta^\beta_\nu \delta^\mu_\alpha. \tag{A.3}$$

We can also see that taking the trace of a single field will give zero:

$$\mathrm{Tr}((T^a)^\beta_\alpha) = \delta^\beta_\nu \delta^\mu_\alpha. \tag{A.4}$$

Appendix B

Quantum mechanic perturbation theory

Perturbation theory is used when we have a well-known Hamiltonian(H_0), to which we add with some small perturbation H_1:

$$H = H_0 + \lambda H_1, \tag{B.1}$$

where λ is some small quantity. The eigenstates and eigenvalues of H_0 are known, and given by:

$$H_0\phi = E_0\phi. \tag{B.2}$$

To find the perturbed eigenstates(ψ) and eigenvalues(E_1), we need to solve the expression:

$$(H_0 + \lambda H_1)\psi = E_1\psi. \tag{B.3}$$

We follow closely the derivation in ref. [15]. Since the eigenstates of H_0 form a complete set, we can expand the perturbed eigenstates in them:

$$\psi_n = M\left(\phi_n + \sum_{k\neq n} C_{nk}(\lambda)\phi_k\right). \tag{B.4}$$

M is set to 1 because of the requirement that for $\lambda \to 0$, $\psi_n \to \phi_n$, and C_{nk} is defined as:

$$C_{nk}(\lambda) = C_{nk}^{(1)}\lambda + C_{nk}^{(2)}\lambda^2. \tag{B.5}$$

The expansion of the eigenstates is defined as:

$$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2. \tag{B.6}$$

Now inserting this expansion in eq. (B.3), first the factors to first order in lambda are isolated to get:

$$E_n^1\phi_n = H_1\phi_n + \sum_{k\neq n}(E_k^0 + E_n^0)C_{nk}^1\phi_k. \tag{B.7}$$

Which gives the following expression for the first correction to the eigenvalue:

$$E_n^1 = \langle\phi_n|H_1|\phi_n\rangle, \tag{B.8}$$

and to the eigenstate:

$$C_{nl}^1 = \frac{\langle\phi_l|H_1|\phi_n\rangle}{E_n^0 + E_l^0}. \tag{B.9}$$

There is no overlap between the double-trace states and the single-trace states, so we need to go to second order in λ for the energy:

$$E_n = E_n + \frac{1}{N^2}\sum_{m\neq l}\frac{\langle\phi_n|H_1|\phi_l\rangle\langle\phi_l|H_1|\phi_n\rangle}{E_n^0 + E_l^0}, \tag{B.10}$$

which gives us the final result for the full eigenstates:

$$\psi_n = \phi_n + \frac{1}{N}\frac{\langle\phi_l|H_1|\phi_n\rangle}{E_n^0 + E_l^0}\phi_n, \tag{B.11}$$

and eigenvalue:

$$E_n = E_n + \frac{1}{N^2}\sum_{m\neq l}\frac{\langle\phi_n|H_1|\phi_l\rangle\langle\phi_l|H_1|\phi_n\rangle}{E_n^0 + E_l^0}. \tag{B.12}$$

Appendix C

Transformations in physics

Lorentz transformations:

$$x^\mu \to x^\mu + w^\mu_\nu x^\nu. \tag{C.1}$$

Translations:

$$x^\mu \to x^\mu + w^\mu_\nu x^\nu. \tag{C.2}$$

Dilatations:

$$x^\mu \to x^\mu + c x^\nu. \tag{C.3}$$

Conformal transformations:

$$x^\mu \to x^\mu + c^\mu x^2 + x^\mu c_\nu x^\nu. \tag{C.4}$$

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